ECE562, Fall 2020: Midterm

1. **Problem 1**

   Let $X_n = X(nT)$ be a discrete-time random process obtained by sampling a real-valued continuous-time mean zero stationary process $X(t)$ with period T.

   (a) Express $r_X(k)$ in terms of $R_X(\tau)$, where $r_X(k)$ and $R_X(\tau)$ are the autocorrelation sequence and function for the discrete- and continuous-time processes, respectively.

   (b) Find $r_X(k)$ if the power density spectrum of $X(t)$ is $S_X(\omega) = 1, 0 \leq |\omega| \leq 2\pi B$ and zero otherwise.

   (c) Determine T such that X_n is a white sequence.

2. **Problem 2**

   Consider the space $\mathcal{S}_N$ of real-valued signals on the interval $[0, 1)$ with the property that each signal is piecewise constant over subintervals of the form $[\frac{k}{N}, \frac{k+1}{N}), k = 0, 1, \ldots, N-1$.

   (a) Find an orthonormal basis $\{\psi_0, \psi_1, \ldots, \psi_{N-1}\}$ for this signal space.

   (b) Let $N = 2$ and consider the constellation $\{(\sqrt{\mathcal{E}}, 0), (0, \sqrt{\mathcal{E}}), (-\sqrt{\mathcal{E}}, 0), (0, -\sqrt{\mathcal{E}})\}$ of 4 signals in $\mathcal{S}_2$, each with energy $\mathcal{E}$. **True or False**: Among all constellations of 4 signals in $\mathcal{S}_2$ with energy of each signal $\leq \mathcal{E}$, the aforementioned constellation results in low probability of error in AWGN ($\mathcal{N}(0, N_0/2)$) assuming equiprobable signals. Justify your answer.

   (c) Compute the minimum distance $d_{\min}$ and bound P_e for the constellation in the previous part assuming equiprobable signals.

3. **Problem 3**

   Consider a system employing antipodal signaling in which the input to the detector is $y_k = A_k + n_k + w_k$, where $A_k \in \{\pm 1\}$ with equal probabilities, $n_k \in \{-\frac{1}{4}, 0, \frac{1}{4}\}$ is a discrete noise random variable (introduced, e.g., by an adversary) such that $P(n_k = -\frac{1}{4}) = P(n_k = \frac{1}{4}) = \frac{3}{8}$ and $P(n_k = 0) = \frac{1}{4}$ and $w_k \sim \mathcal{N}(0, 1)$ is Gaussian noise. Suppose that A_k, n_k, w_k are independent. Express P_e in terms of the Q-function for a detector with decision threshold $\tau = 0$.

4. **Problem 4**

   Consider a modulation scheme in which blocks of ν bits determine a modulated waveform of the form $\mathrm{Re}\{A_i e^{j\theta_i} e^{j2\pi f t}\}$ on $[0, T]$, where $f = 1/T$. The first κ bits determine $\theta_i \in \mathbb{R}$ and the remaining $\nu - \kappa$ bits determine $A_i \in \mathbb{R}$.

   (a) Find an orthonormal basis for the modulated waveforms.

   (b) Find the dimension of the space spanned by this basis.