ECE 562 Advanced Digital Communications **University of Illinois at Urbana-Champaign**
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HW4 Solution

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Problem 1

(a) By the hints provided during office hours,

$$X''_{rc}(f) = \begin{cases} -\frac{T^3\pi^2}{2\beta^2}\cos\left(\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right), & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T}, \\ 0, & \text{otherwise} \end{cases}$$

or

$$X''_{rc}(f) = -\frac{T^2\pi^2}{\beta^2}\left[X_{rc}(f) - \frac{T}{2}\left(u\left(f + \frac{1-\beta}{2T}\right) - u\left(f - \frac{1-\beta}{2T}\right) + u\left(f + \frac{1+\beta}{2T}\right) - u\left(f - \frac{1+\beta}{2T}\right)\right)\right].$$

Using now the pair $\mathcal{F}\left\{\frac{d^n}{dt^n}x(t)\right\} = (\jmath 2\pi f)^n X(f)$ and the dual to the derivative theorem of the Fourier transform $\mathcal{F}\{-\jmath 2\pi t x(t)\} = X'(f)$, the inverse Fourier transform to both sides of the last equation yields

$$-4\pi^2 t^2 x_{rc}(t) = -\frac{T^2\pi^2}{\beta^2}\left[x_{rc}(t) - \frac{T}{2}\frac{1}{\pi t}\left(\sin\left(\frac{1-\beta}{2T}2\pi t\right) + \sin\left(\frac{1+\beta}{2T}2\pi t\right)\right)\right].$$

Solving for $x_{rc}(t)$ the result follows.

(b) We first note that

$$\frac{\sin(\pi t/T)}{\pi t/T} = 1$$

for $t = 0$ and 0 for $t = nT$. Also,

$$\frac{\cos(\beta\pi t/T)}{1 - 4\beta^2 t^2/T^2} = 1$$

for $t = 0$ and $< \infty$ for $t \neq 0$ (use of L'Hôpital's rule for values of t such that $4\beta^2 t^2/T^2 = 1$). Therefore,

$$x_{rc}(nT) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

(c)

$$\begin{aligned} X_{rc}(f) &= \int_{-\frac{1+\beta}{2T}}^{-\frac{1-\beta}{2T}} \frac{T}{2}\left[1 + \cos\left(\frac{\pi T}{\beta}\left(-f - \frac{1-\beta}{2T}\right)\right)\right] df \\ &+ \int_{-\frac{1-\beta}{2T}}^{\frac{1-\beta}{2T}} T df + \int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} \frac{T}{2}\left[1 + \cos\left(\frac{\pi T}{\beta}\left(f - \frac{1-\beta}{2T}\right)\right)\right] df \\ &= 1 + \frac{T}{2}\left[\int_{-\frac{1+\beta}{2T}}^{-\frac{1-\beta}{2T}} \cos\left(\frac{\pi T}{\beta}\left(f + \frac{1-\beta}{2T}\right)\right) df + \int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} \cos\left(\frac{\pi T}{\beta}\left(f - \frac{1-\beta}{2T}\right)\right) df\right] \\ &= 1, \end{aligned}$$

where the even symmetry of $\cos(x)$ has been used and the two straighforward integrations have been evaluated.

Problem 2

(a) Clearly, $h(t) = \mathcal{F}^{-1}\{H(f)\} = \delta(t) + \frac{a}{2}[\delta(t - t_0) + \delta(t + t_0)]$ and the result follows.

(b) Let $x(t) = \sum_n c_n s(t - nT)$. Then $y(t) = x(t) * h(t) + w(t)$ and the result should be explicit based on the discussion during office hours.

Problem 3

Based on the hints provided during office hours, we have that

(a) $\mathbf{C} = \mathbf{H}^+$, i.e., the Moore-Penrose pseudoinverse of $\mathbf{H}$.

(b) Let the SVD of the channel matrix be $\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\dagger$ where $\mathbf{U} \in \mathbb{C}^{n_R \times n_R}, \mathbf{V} \in \mathbb{C}^{n_T \times n_T}$ are unitary matrices and $\mathbf{\Sigma} \in \mathbb{R}^{n_R \times n_T}$ contains in the main diagonal the singular values of $\mathbf{H}$ and zeros elsewhere. Let the transmit precoding scheme correspond to transmitting $\mathbf{x} = \mathbf{V}\tilde{\mathbf{x}}$ and the receiver shaping scheme correspond to linearly preprocessing the incoming signal $\mathbf{y}$ to form $\tilde{\mathbf{y}} = \mathbf{U}^\dagger\mathbf{y}$. Then, the effective channel model becomes

$$\tilde{\mathbf{y}} = \mathbf{U}^\dagger\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\dagger\mathbf{V}\tilde{\mathbf{x}} + \mathbf{U}^\dagger\mathbf{w} = \mathbf{\Sigma}\tilde{\mathbf{x}} + \tilde{\mathbf{w}},$$

where $\tilde{\mathbf{w}} = \mathbf{U}^\dagger\mathbf{w}$. This corresponds to effectively turning the MIMO channel into $\mathrm{rank}(\mathbf{H})$ parallel SISO subchannels. The noise statistics are invariant to unitary transformations.

(c) (i) $\hat{x} = \mathbf{c}^\dagger\mathbf{y} = \mathbf{c}^\dagger\mathbf{h}x + \mathbf{c}^\dagger\mathbf{w}$. Therefore,

$$\mathrm{SNR} = \frac{|\mathbf{c}^\dagger\mathbf{h}|^2}{E[|\mathbf{c}^\dagger\mathbf{w}|^2]} = \frac{|\mathbf{c}^\dagger\mathbf{h}|^2}{\|\mathbf{c}\|^2} \le \frac{\|\mathbf{c}\|^2\|\mathbf{h}\|^2}{\|\mathbf{c}\|^2},$$

where the last inequality is due to the Cauchy-Schwarz inequality. Equality is achieved when $\mathbf{c}_*$ is collinear to $\mathbf{h}$. This is referred to as *matched filter* or *maximum ratio combining*. The achieved SNR is $\|\mathbf{h}\|^2$.

(ii) $y = \mathbf{h}^\dagger\mathbf{x} + w = \mathbf{h}^\dagger\mathbf{s}x + w$. The beamvector corresponds to the optimizing solution of the problem

$$\max_{\mathbf{s}} \ \mathrm{SNR} = E[|\mathbf{h}^\dagger\mathbf{s}x|^2] = |\mathbf{h}^\dagger\mathbf{s}|^2 \quad \text{such that} \ \ \|\mathbf{s}\|^2 \le p_0.$$

By the Cauchy-Schwarz inequality and the power constraint, $\mathrm{SNR} \le \|\mathbf{h}\|^2\|\mathbf{s}\|^2 \le \|\mathbf{h}\|^2 p_0$. Both equalities are achieved when $\mathbf{s}_* = \sqrt{p_0}\frac{\mathbf{h}}{\|\mathbf{h}\|}$ and the achieved SNR is $\|\mathbf{h}\|^2 p_0$.