ECE586 MDPs and Reinforcement Learning **University of Illinois at Urbana-Champaign**
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Lecture 1: Markov Chains-Part I

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1.1 Definition and characterization

In this section we give the definition of a Markov Chain and several related concepts.

A *stochastic process* is an indexed collection of random variables $X = \{X(t) : t \in T\}$ or $X = \{X_t : t \in T\}$ with T being the index set, all of them defined on the same probability space $(\Omega, \mathcal{F}, P)$. t often represents time, in which case $X(t)$ or X_t is the state at time t, which models the time variation of a particular quantity. If $T = \mathbb{R}$ or T is an interval of $\mathbb{R}$, then X is a continuous-time random process. If T is a countable set, then X is a discrete-time process. For example, if $T = \mathbb{N}$, then $X = \{X_0, X_1, X_2, \cdots\}$. For $T = \mathbb{N}$ or $T = \mathbb{Z}$ we may alternatively write $X = \{X_n\}_{n\geq 0}$ or $X = \{X_n\}_{n\in\mathbb{Z}}$, respectively.

Suppose $X_k \in \mathcal{X}$ for all k. The set $\mathcal{X}$ is called *state space*.

Definition 1.1. *(Markov Chain) A discrete-time stochastic process $X = \{X_0, X_1, X_2, \cdots\}$ with a countable state space $\mathcal{X}$ is a Markov Chain if*

$$P\left(X_{k+1} = j | X_k = i, X_{k-1} = i_{k-1}, \cdots, X_0 = i_0\right) = P(X_{k+1} = j | X_k = i) \quad \forall i, j, i_{k-1}, \cdots, i_0 \ \ \textit{and} \ \ \forall k \geq 0. \tag{1.1}$$

In words, a Markov Chain is a stochastic process such that given the value of the current state, the distribution of the next state is independent of the past. (1.1) is called *Markov property*.

Note: (1.1) requires that the conditional probabilities in both sides are well-defined, i.e., $P(X_k = i, X_{k-1} = i_{k-1}, \cdots, X_0 = i_0) > 0$.

A Markov Chain is *time-homogeneous* if the transition probabilities $P(X_{k+1} = j | X_k = i)$ do not depend on k, in which case we denote them as P_{ij}. We will assume finite state time-homogeneous Markov Chains from now on. The matrix $P = [P_{ij}]_{i,j\in\mathcal{X}}$ is called the *transition matrix* of the Markov Chain X. P satisfies the properties that $P_{ij} \geq 0$ for all $i, j \in \mathcal{X}$ and $\sum_{j\in\mathcal{X}} P_{ij} = 1$ for all $i \in \mathcal{X}$, i.e. the entries in each row of P sum to one. A matrix with this property is called *stochastic* (or sometimes row-stochastic).

Let $p(k)$ be a row vector with the ith entry being $p_i(k) \triangleq P(X_k = i)$, the probability that the value of the state at time k equals i. Then,

$$p(k) = p(k-1)P. \tag{1.2}$$

The above equation can be easily checked by noting that $p_j(k) = P(X_k = j) = \sum_{i\in\mathcal{X}} p_i(k-1)P_{ij}$, where the last step is by the law of total probability. By iterating Equation (1.2) k times, we obtain:

$$p(k) = p(0)P^k. \tag{1.3}$$

We call P^k the *k-step transition matrix*. The interpretation of P^k is that its entries correspond to the k-step transition probabilities $P_{ij}^{(k)} = P(X_{n+k} = j | X_n = i)$ for any n. That is, $P_{ij}^{(k)}$ is the probability that we start from state i (at time n) and arrive at state j in k steps. From (1.3), we see that the state probabilities at any time k are completely characterized by P and $p(0)$. In fact, the distribution or *probability law* of a discrete-time homogeneous Markov Chain is determined by its initial distribution and its transition matrix since for any $k \geq 0$ and any $i_0, i_1, \dots i_k$:

$$P(X_0 = i_0, X_1 = i_1, \dots, X_k = i_k) = p_{i_0}(0) P_{i_0 i_1} \cdots P_{i_{k-1} i_k}.$$

Finally, we note here that for any i, j and $n, m \geq 0$ the **Chapman-Kolmogorov equations** hold:

$$P_{ij}^{(n+m)} = \sum_{k \in \mathcal{X}} P_{ik}^{(n)} P_{kj}^{(m)}.$$

These equations provide an alternative justification of the fact that the k-step transition matrix $P^{(k)} = [P_{ij}^{(k)}]$ coincides with P^k (i.e., P raised to the power k). To see this, use the Chapman-Kolmogorov equations to write $P^{(k)} = P^{(k-1)} P^{(1)} = P^{(k-1)} P$ and iterate this relationship to obtain $P^{(k)} = P^k$.

Note: Due to $P^{(k)} = P^k$ as a consequence of the Chapman-Kolmogorov equations, sometimes in the literature (1.3) (and by extension (1.2)) is called Chapman-Kolmogorov equation.

Moreover, a Markov Chain can be represented by a *transition graph*.

Example. For a Markov Chain with transition matrix $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$, the corresponding transition graph is drawn in Figure 1.1.

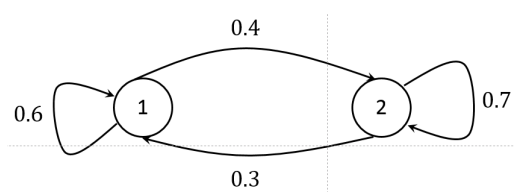

Figure 1.1: An example of a Markov Chain transition graph.

From the example, we see that the transition graph is a directed graph and an arrow from node i to j exists when $P_{ij} > 0$.

Lastly, many homogeneous Markov Chains have a natural representation in terms of a recurrence equation driven by white noise, which is more concretely explained by the following theorem.

Theorem 1.2. *Let $\{\xi_n\}_{n \geq 1}$ be an i.i.d. sequence of random variables with values in some arbitrary space E. Let $\mathcal{X}$ be a countable space and $f : \mathcal{X} \times E \to \mathcal{X}$ be some function. Let also X_0 be a random variable with values in $\mathcal{X}$, independent of $\{\xi_n\}_{n \geq 1}$. Then the recurrence $X_{n+1} = f(X_n, \xi_{n+1})$ defines a homogeneous Markov Chain.*

Example. (1-D random walk on $\mathbb{Z}$) Suppose $X_{n+1} = X_n + \xi_{n+1}$, with X_0 being an integer and $\{\xi_n\}$ is a sequence of independent Radamacher random variables, independent of X_0. That is,

$$\xi_n = \begin{cases} +1 & w.p. \quad 1/2 \\ -1 & w.p. \quad 1/2 \end{cases}.$$

More generally, ξ_n can take the value $+1$ with probability p and the value -1 with probability $1 - p$. Then $X = \{X_n\}_{n \geq 0}$ is a homogeneous Markov Chain by the previous theorem and is called *random walk on $\mathbb{Z}$*.

Note: Not all homogeneous Markov Chains are naturally described by this model (although Discrete-Time Markov Chains (DTMC) that are homogeneous can always be represented *at least distributionwise* by a stochastic recurrence equation $X_{n+1} = f(X_n, \xi_{n+1})$, where $\{\xi_n\}_{n \geq 1}$ is an i.i.d. sequence, independent of X_0).

1.2 Communicating class, transience and recurrence, irreducibility

In this section we discuss some properties of the states in a Markov Chain.

A state j is *reachable* (or *accessible*) from state i if there exists a path from i to j in the transition graph $i \to i_1 \to i_2 \to \cdots \to j$. This implies that there exists an integer $n_{ij} \geq 1$ such that $P_{ij}^{(n_{ij})} > 0$. A node, by definition, is said to be always reachable from itself in a Markov Chain.

Definition 1.3. *(Communicating Class) A set C of states is a* communicating class *if for all $i, j \in C$, j is reachable from i and i is reachable from j, i.e., every pair of states in C* ***communicates with each other****. A communicating class C is* maximal*, i.e., adding a state to C would mean that not all states in C are reachable from each other anymore.*

Definition 1.4. *(Transience and Recurrence) A state i is called* transient *if starting at state i there is a nonzero probability of never returning to i. A state i is called* recurrent *if it is not transient.*

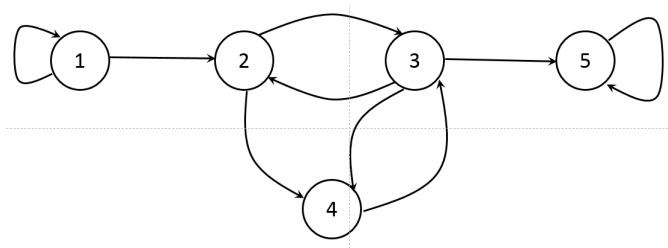

Figure 1.2: An example of a Markov Chain with three communicating classes

Example. Figure 1.2 shows a Markov Chain with three communicating classes: $\{1\}$, $\{2, 3, 4\}$ and $\{5\}$. State 1 is transient. State 5 is an *absorbing* state since it is impossible to leave it.

Recurrence and transience are class properties, i.e. they either hold or do not hold for all states in a communicating class. In the previous example, all states in $\{2, 3, 4\}$ are transient.

Definition 1.5. *(Irreducibility) If a Markov Chain has a single communicating class, we say that it is* irreducible.

1.2.1 Stationary and limiting distributions

Theorem 1.6. *A finite state irreducible Markov Chain has a* unique *stationary measure (otherwise known as a stationary or invariant or equilibrium probability distribution). This stationary measure, π, is a row probability vector such that*

$$\pi = \pi P \,. \tag{1.4}$$

Theorem (1.6) implies that if X_0 is drawn from π (we denote this by $X_0 \sim \pi$), then $X_n \sim \pi$ for all n.

We note that because P is a stochastic matrix, it always has an eigenvalue equal to 1. This can be easily seen by noting that $P1 = 1$, where 1 is the all-ones vector of appropriate dimension. Therefore, 1 is an eigenvalue of P and by extension of P^T, i.e., of the transpose of P. This implies that π is the normalized left eigenvector of P corresponding to eigenvalue 1.

Example. Figure 1.3 represents a two-state Markov Chain with probability matrix $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Is this Chain irreducible?

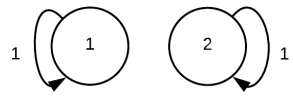

Figure 1.3: An example of a Markov Chain with two states

Answer: No. The Markov Chain has more than one communicating class. Also, if it were irreducible then there would exist a unique stationary distribution. However, for the given P, we have that $\pi = \pi P$ for all π (since P is the identity matrix).

Definition 1.7. *(Limiting Distribution) The* limiting distribution *of a Markov Chain is:*

$$\lim_{k\to\infty} p(k) = \lim_{k\to\infty} p(0)P^k\,.$$

The limiting distribution may not exist.

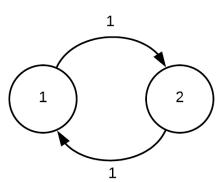

Figure 1.4: An example of a Markov Chain with two states and no limiting distribution for $p(0) \neq (1/2\ \ 1/2)$.

Example. Figure 1.4 represents a two-state Markov Chain with probability matrix $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Suppose that $p(0) = (1\ \ 0)$, i.e. the Markov Chain starts in state 1 with probability 1. Running the $p(k)$ recursion, we obtain:

$$p(k) = \begin{cases} (0\ \ 1), & \text{if } k \text{ is odd} \\ (1\ \ 0), & \text{if } k \text{ is even}\,. \end{cases}$$

Thus we see that the sequence $\{p(k)\}$ oscillates and therefore, it does not have a limiting distribution.

By the definition of the limiting distribution, the associated initial distribution $p(0)$ is important. As we see above, for the initial distribution $p(0) = (1\ \ 0)$, the limiting distribution does not exist. However, if $p(0) = (1/2\ \ 1/2)$, then $p(k) = (1/2\ \ 1/2)$ for any k. Thus, in this particular case, a limiting distribution exists. Additionally, since in the long-run the Markov Chain will spend half of the time in each state, it is trivial to see that the stationary distribution π for this chain is $(1/2\ \ 1/2)$ (can be also derived by solving $\pi = \pi P$ and by taking into account that $\sum_{i\in\mathcal{X}} \pi_i = 1$). Therefore, for $p(0) = (1/2\ \ 1/2)$:

$$\lim_{k\to\infty} p(k) = \pi.$$

The limiting distributions (if they exist) are always a *subset* of the stationary distributions. Therefore, any limiting distribution is a stationary distribution. The converse is not true in general.

Definition 1.8. *(Regular transition matrix) A transition matrix P is* regular *(or primitive) if there exists a positive integer k such that $P_{ij}^{(k)} > 0\ \forall\, i, j$. Recall here that $P_{ij}^{(k)}$ is the (i, j)th element of $P^{(k)}$, which coincides with P^k.*

Regularity is stronger than irreducibility.

Example. If $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then for any integer k any two entries of P^k are 0. To see this, note that for all $n \in \mathbb{N}$, $P^{2n} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $P^{2n+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P$. Thus this Markov Chain is not regular, but it is irreducible.

If P is regular, then there exists a unique stationary distribution.