ECE562, Fall 2020: Problem Set #4

Due Nov 20, 2020

1. **Intersymbol Interference and Pulse Shaping**

   The Fourier transform of the raised cosine pulse is given by

$$X_{rc}(f) = \begin{cases} T, & 0 \le |f| \le \frac{1-\beta}{2T}, \\ \frac{T}{2}\left[1 + \cos\left(\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right)\right], & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0, & |f| > \frac{1+\beta}{2T} \end{cases},$$

   where $\beta \in [0, 1]$.

   (a) Show that the time domain expression of this pulse is

$$x_{rc}(t) = \frac{\sin(\pi t/T)}{\pi t/T}\frac{\cos(\beta\pi t/T)}{1 - 4\beta^2 t^2/T^2}$$

   (b) Prove that for any value of the rolloff factor β the raised cosine pulse satisfies the Nyquist Criterion for zero ISI.

   (c) Show that

$$\int_{-\infty}^{\infty} X_{rc}(f)df = 1$$

2. **More on ISI**

   Consider a channel with frequency response $H(f) = 1 + a\cos(2\pi f t_0), |a| < 1, |f| \le F_{\max}$ and 0 otherwise. An input signal $s(t)$, bandlimited to $F_{\max}$, is transmitted through this channel.

   (a) Show that $y(t) = s(t) + \frac{a}{2}[s(t - t_0) + s(t + t_0)]$.

   (b) Determine the output of a filter matched to $s(t)$ at $t = kT, k \in \mathbb{Z}$ when the applied input is $y(t)$. Here, T is the symbol duration.

3. **Introduction to MIMO channels**

   MIMO channels arise, e.g., in wireline or multi-antenna wireless systems and are characterized by the existence of multiple communication dimensions, which enhance *spectral efficiency* and *link reliability*. Due to the aforementioned multiple dimensions, MIMO channels have the intrinsic *multiplexing* property, which allows for the transmission of several symbols simultaneously. More specifically, the higher dimensionality of MIMO channels allows for the establishment of multiple parallel subchannels, also called *channel eigenmodes*, which result in a rate increase.

   Consider a point-to-point communication system with n_T transmit and n_R receive antennas. The input-output relation for an uncoded MIMO transmission, when the channel is flat in frequency, is given by the baseband signal model

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w} \quad (1)$$

Here, $\mathbf{x}$ is the $n_T \times 1$ complex-valued transmit vector, $\mathbf{H} = [h_{ij}]$ is an $n_R \times n_T$ matrix of complex channel gains with h_{ij} being the channel gain or *fading* from the jth transmit to the ith receive antenna and $\mathbf{w}$ is a spatially and temporally white mean-zero complex circularly symmetric Gaussian noise vector with a scaled identity covariance matrix. More generally, the channel will be frequency-selective and be described by the matrix convolution

$$\mathbf{y}(n) = \sum_{k=0}^{L} \mathbf{H}(k)\mathbf{x}(n-k) + \mathbf{w}(n) \tag{2}$$

Focusing on (1), the ML detector decides

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2$$

Note that $\mathbf{x}$ takes values over a set of cardinality $2^{\nu n_T}$, where 2^{ν} is the size of the modulation scheme used and ν is the number of bits per signal. This implies that the complexity of the ML detector grows exponentially in (ν, n_T). Clearly, the applicability of ML detection is only possible when the employed signal constellations have small sizes and the number of transmit antennas is also small. The *sphere detection algorithm* is a reduced-complexity approximation of the ML detector. The underlying idea is to search for the optimum $\mathbf{x}$ into a smaller subset of high probability, which is defined by a hyperball centered at $\mathbf{y}$ and having radius r, i.e., in a set of the form $\{\mathbf{x} | \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2 \leq r^2\}$.

(a) (**zero-forcing receiver**) A different approach to reduce the receiver complexity, at the price of performance loss, is to consider a linear receiver preprocessing $\mathbf{y}$ linearly to produce $\tilde{\mathbf{y}} = \mathbf{C}\mathbf{y}$. Suppose that $\mathbf{C}$ is chosen to remove the off-diagonal entries of $\mathbf{CH}$. This allows for coordinate detection, i.e., separate detection of each entry in $\mathbf{x}$. Determine $\mathbf{C}$.

(b) (**parallel decomposition of the channel**) Suppose that the channel matrix $\mathbf{H}$ is known to both the transmitter and the receiver. In a *rich scattering environment*, $\mathbf{H}$ is full rank, i.e., $\text{rank}(\mathbf{H}) = \min\{n_T, n_R\}$. Nevertheless, $\mathbf{H}$ can be a rank-one matrix if there is high correlation among its gain elements. Describe a transmit precoding and receiver shaping scheme turning the MIMO channel into rank($\mathbf{H}$) SISO channels. Do the noise statistics change?

(c) (**Beamforming**) *Beamforming* is associated with array processing and conceptually related to *linear MIMO transceivers*.

   i. (**receive beamforming-maximum ratio combining**) Consider a SIMO channel $\mathbf{y} = \mathbf{h}x + \mathbf{w}$, where $E[|x|^2] = 1, E[\mathbf{w}] = \mathbf{0}$ and $E[\mathbf{w}\mathbf{w}^\dagger] = \mathbf{I}$. Receive beamforming is used to estimate x by using a beamvector $\mathbf{c}$ as $\hat{x} = \mathbf{c}^\dagger \mathbf{y}$. Determine the beamvector maximizing the SNR and the achieved SNR.

   ii. (**transmit beamforming**) Consider a MISO channel $y = \mathbf{h}^\dagger \mathbf{x} + w$, where $E[w] = 0, E[|w|^2] = 1$. Employing transmit beamforming, a weighted version of the symbol x via the beamvector $\mathbf{s}$ is transmitted on each antenna, i.e., $\mathbf{x} = \mathbf{s}x$. Let $E[|x|^2] = 1$ and impose on the beamvector $\mathbf{s}$ the power constraint $\|\mathbf{s}\|^2 \leq p_0$. Determine $\mathbf{s}$ maximizing the SNR and the achieved SNR.