1. **Intersymbol Interference and Pulse Shaping**

The Fourier transform of the raised cosine pulse is given by

\[
X_{rc}(f) = \begin{cases} 
T, & 0 \leq |f| \leq 1 - \frac{\beta}{2T}, \\
\frac{T}{2} \left[ 1 + \cos \left( \frac{\pi T}{\beta} \left( |f| - \frac{1 - \beta^2}{2T} \right) \right) \right], & 1 - \frac{\beta}{2T} \leq |f| \leq 1 + \frac{\beta}{2T}, \\
0, & |f| > 1 + \frac{\beta}{2T}
\end{cases}
\]

where \( \beta \in [0, 1] \).

(a) Show that the time domain expression of this pulse is

\[
x_{rc}(t) = \frac{\sin(\pi t/T)}{\pi t/T} \cos(\beta \pi t/T)
\]

(b) Prove that for any value of the rolloff factor \( \beta \) the raised cosine pulse satisfies the Nyquist Criterion for zero ISI.

(c) Show that

\[
\int_{-\infty}^{\infty} X_{rc}(f) df = 1
\]

2. **More on ISI**

Consider a channel with frequency response \( H(f) = 1 + a \cos(2\pi ft_0), |a| < 1, |f| \leq F_{max} \) and 0 otherwise. An input signal \( s(t) \), bandlimited to \( F_{max} \), is transmitted through this channel.

(a) Show that \( y(t) = s(t) + \frac{a}{2}[s(t - t_0) + s(t + t_0)] \).

(b) Determine the output of a filter matched to \( s(t) \) at \( t = kT, k \in \mathbb{Z} \) when the applied input is \( y(t) \). Here, \( T \) is the symbol duration.

3. **Introduction to MIMO channels**

MIMO channels arise, e.g., in wireline or multi-antenna wireless systems and are characterized by the existence of multiple communication dimensions, which enhance spectral efficiency and link reliability. Due to the aforementioned multiple dimensions, MIMO channels have the intrinsic multiplexing property, which allows for the transmission of several symbols simultaneously. More specifically, the higher dimensionality of MIMO channels allows for the establishment of multiple parallel subchannels, also called channel eigenmodes, which result in a rate increase.

Consider a point-to-point communication system with \( n_T \) transmit and \( n_R \) receive antennas. The input-output relation for an uncoded MIMO transmission, when the channel is flat in frequency, is given by the baseband signal model

\[
y = \mathbf{Hx} + \mathbf{w}
\]
Here, \( x \) is the \( n_T \times 1 \) complex-valued transmit vector, \( H = [h_{ij}] \) is an \( n_R \times n_T \) matrix of complex channel gains with \( h_{ij} \) being the channel gain or fading from the \( j \)th transmit to the \( i \)th receive antenna and \( w \) is a spatially and temporally white mean-zero complex circularly symmetric Gaussian noise vector with a scaled identity covariance matrix. More generally, the channel will be frequency-selective and be described by the matrix convolution

\[
y(n) = \sum_{k=0}^{L} H(k)x(n - k) + w(n)
\]  

(2)

Focusing on (1), the ML detector decides

\[
\hat{x} = \arg\min_x \|y - Hx\|^2
\]

Note that \( x \) takes values over a set of cardinality \( 2^{\nu n_T} \), where \( 2^\nu \) is the size of the modulation scheme used and \( \nu \) is the number of bits per signal. This implies that the complexity of the ML detector grows exponentially in \( (\nu, n_T) \). Clearly, the applicability of ML detection is only possible when the employed signal constellations have small sizes and the number of transmit antennas is also small. The sphere detection algorithm is a reduced-complexity approximation of the ML detector. The underlying idea is to search for the optimum \( x \) into a smaller subset of high probability, which is defined by a hyperball centered at \( y \) and having radius \( r \), i.e., in a set of the form \( \{x|\|y - Hx\|^2 \leq r^2\} \).

(a) (zero-forcing receiver) A different approach to reduce the receiver complexity, at the price of performance loss, is to consider a linear receiver preprocessing \( y \) linearly to produce \( \tilde{y} = Cy \). Suppose that \( C \) is chosen to remove the off-diagonal entries of \( CH \). This allows for coordinate detection, i.e., separate detection of each entry in \( x \). Determine \( C \).

(b) (parallel decomposition of the channel) Suppose that the channel matrix \( H \) is known to both the transmitter and the receiver. In a rich scattering environment, \( H \) is full rank, i.e., \( \text{rank}(H) = \min\{n_T, n_R\} \). Nevertheless, \( H \) can be a rank-one matrix if there is high correlation among its gain elements. Describe a transmit precoding and receiver shaping scheme turning the MIMO channel into rank(\( H \)) SISO channels. Do the noise statistics change?

(c) (Beamforming) Beamforming is associated with array processing and conceptually related to linear MIMO transceivers.

i. (receive beamforming-maximum ratio combining) Consider a SIMO channel \( y = hx + w \), where \( E[|x|^2] = 1, E[w] = 0 \) and \( E[ww^\dagger] = I \). Receive beamforming is used to estimate \( x \) by using a beamvector \( c \) as \( \hat{x} = c^\dagger y \). Determine the beamvector maximizing the SNR and the achieved SNR.

ii. (transmit beamforming) Consider a MISO channel \( y = h^\dagger x + w \), where \( E[w] = 0, E[|w|^2] = 1 \). Employing transmit beamforming, a weighted version of the symbol \( x \) via the beamvector \( s \) is transmitted on each antenna, i.e., \( x = sx \). Let \( E[|x|^2] = 1 \) and impose on the beamvector \( s \) the power constraint \( \|s\|^2 \leq p_0 \). Determine \( s \) maximizing the SNR and the achieved SNR.