ECE562, Fall 2020: Problem Set #3

Due Oct 30, 2020

1. **Matched Filter**
   Let $r(t) = s(t) + w(t)$ be the received signal and assume that $s(t)$ is a (real-valued) signal with energy $\mathcal{E}$ and $w(t)$ is mean zero AWGN with power spectral density $N_0/2$. Consider an LTI filter with impulse response $h(t)$ followed by a sampler sampling the filter's output at $t = T$. Let $r(t)$ be the input to the aforementioned filter. The corresponding output SNR is given by

   $$\mathrm{SNR}_o = \frac{[(h * s)(T)]^2}{E[((h * w)(T))^2]}.$$

   (a) Use frequency domain arguments to show that $\mathrm{SNR}_o \leq \frac{2\mathcal{E}}{N_0}$.

   (b) By the proof of part (a) conclude that the frequency response of the filter maximizing the output SNR is $S^*(\omega)e^{-\jmath\omega T}$ and use Fourier transform properties to show that the optimum filter is matched to $s(t)$.

   Note: *Cauchy-Schwarz inequality for the inner product space of square-integrable complex-valued functions*: $|\int_{-\infty}^{\infty} X_1(\omega)X_2^*(\omega)d\omega|^2 \leq \int_{-\infty}^{\infty} |X_1(\omega)|^2 d\omega \int_{-\infty}^{\infty} |X_2(\omega)|^2 d\omega$.

2. **Matched Filter for Colored Noise (Prewhitening)**

   Consider Problem 1 and assume that the noise is colored with power spectrum $S_w(\omega)$.

   (a) Show that $\mathrm{SNR}_o \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|S(\omega)|^2}{S_w(\omega)} d\omega$.

   (b) Prove that the frequency response of the optimum filter, called *generalized matched filter*, is proportional to $\frac{S^*(\omega)e^{-\jmath\omega T}}{S_w(\omega)}$. Provide an argument showing that the generalized matched filter can be decoupled into a *prewhitening filter*, which flattens the power spectral density of the input colored noise, followed by a filter matched to the prefiltered signal.

3. **Error Probability in Hypercube Signal Spaces**
   Consider a signal space $\mathcal{L}$ such that $\dim(\mathcal{L}) = N$ and associate with this space an $M-$ary communication system with $M = 2^N$. Suppose that the signal vectors correspond to the vertices of a hypercube centered at the origin. Determine the average probability of symbol error in terms of $\mathcal{E}_s/N_0$, assuming that all signals are equiprobable. Here, $\mathcal{E}_s$ is the energy per symbol and $N_0/2$ is the power spectral density of the AWGN.

   Note: Translation invariance of the probability of error for transmission over an AWGN channel under ML detection, that has been discussed in class, as well as the symmetry of the constellation may be used to simplify the derivation.

4. **Noncoherent Detection in On-Off Keying**
   Consider a system with the following two possible signals

   $$s_0(t) = 0, \quad 0 \leq t \leq T, \quad s_1(t) = \sqrt{\frac{2\mathcal{E}}{T}}\cos(2\pi f_c t), \quad 0 \leq t \leq T.$$

The corresponding received signals are

$$r(t) = w(t), \quad 0 \le t \le T, \quad r(t) = \sqrt{\frac{2\mathcal{E}}{T}} \cos(2\pi f_c t + \theta) + w(t), \quad 0 \le t \le T$$

where θ is the carrier phase and $w(t)$ is AWGN.

(a) Sketch the block diagram of the receiver (i.e., demodulator and detector) performing noncoherent detection.

(b) Determine the pdfs of the decision variables at the detector for the two possible received signals.

5. **Probability of Error for M-ary Orthogonal Signals**
Show that the average probability of a symbol error for M-ary orthogonal signaling with equal priors in AWGN can be computed as

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[1 - (1 - Q(x))^{M-1}\right] e^{-\frac{\left(x - \sqrt{\frac{2\mathcal{E}_s}{N_0}}\right)^2}{2}} dx.$$

6. **Union Bound and Shannon Limit in Problem 5**

Derive a union bound for P_e in Problem 5 and use it to conclude that $P_e \to 0$ as $M \to \infty$ at an exponential rate, provided that

$$\frac{\mathcal{E}_b}{N_0} > 2 \ln 2.$$

Use a more elaborate bounding argument to improve the last bound to the *Shannon limit*

$$\frac{\mathcal{E}_b}{N_0} > \ln 2.$$

Hint: You may use the Union Bound Theorem for the ML detector on an AWGN channel discussed in class. The Chernoff bound of the Q-function $Q(x) \le e^{-\frac{x^2}{2}}, x > 0$ may be useful.

Note: Hints for all problems will be provided during office hours.