Problem 1

Since $X \sim \mathcal{N}(0, \sigma^2 I_n)$, $Y \sim \mathcal{N}(m_Y, \Sigma_Y)$ with $m_Y = E[AX] = AE[X] = 0$ and $\Sigma_Y = E[(Y - m_Y)(Y - m_Y)^T] = A\Sigma_X A^T = \sigma^2 A A^T$. Clearly, $X, Y$ are identically distributed when $\Sigma_Y = \Sigma_X$, i.e., $AA^T = I_n$. In other words, $X, Y$ are identically distributed when $A$ is an orthogonal matrix.

Problem 2

By the hint, $E[X_1X_2X_3X_4] = (\ldots)$. Moreover, for a (degenerate or non-degenerate) Gaussian random vector $X$,

$$\phi_X(u) = E[e^{u^TX}] = e^{u^TM_X - \frac{1}{2}u^T\Sigma_X u}$$

and since $X \sim \mathcal{N}(0, \Sigma)$ we have that $\phi_X(u) = e^{-\frac{1}{2}u^T\Sigma u} = e^{-\frac{1}{2}\sum_{i,j=1}^n u_i\Sigma_{ij}u_j}$. Clearly,

$$\frac{\partial \phi_X(u)}{\partial u_k} = e^{-\frac{1}{2}\sum_{i,j=1}^n u_i\Sigma_{ij}u_j} \left[ -\frac{1}{2} \left( \sum_{j=1}^n \Sigma_{kj}u_j + \sum_{i=1}^n u_i\Sigma_{ik} \right) \right]$$

$$= e^{-\frac{1}{2}\sum_{i,j=1}^n u_i\Sigma_{ij}u_j} \left[ -\frac{1}{2} \left( \sum_{j=1}^n \Sigma_{kj}u_j + \sum_{i=1}^n u_i\Sigma_{ik} \right) \right]$$

$$= -\sigma_k^T u e^{-\frac{1}{2}\sum_{i,j=1}^n u_i\Sigma_{ij}u_j}. \quad (1.1)$$

Here, $\Sigma_{ik} = \Sigma_{kj}$ has been used and $\sigma_k^T$ denotes the $k$th row of $\Sigma$. Furthermore,

$$\frac{\partial \sigma_k^T u}{\partial u_m} = \frac{\partial}{\partial u_m} \left( \sum_{j=1}^n \Sigma_{kj}u_j \right) = \Sigma_{km} = \Sigma_{mk}. \quad (1.2)$$

Performing the complete differentiation for $E[X_1X_2X_3X_4]$ by employing (1.1) and (1.2) the result follows.

Problem 3

By the definition of the autocorrelation function $R_Y(t+\tau, t) = E[Y(t+\tau)Y(t)] = E[X(t^2+\tau)X(t^2)]$. By employing the property in Problem 2 for $X_1 = X(t+\tau), X_2 = X(t+\tau), X_3 = X(t), X_4 = X(t)$ we obtain

$$R_Y(t+\tau, t) = R_X(t+\tau, t+\tau)R_X(t, t) + 2R_X^2(t, t).$$

For a mean-zero stationary Gaussian process, this equation takes the form $R_Y(\tau) = R_X^2(0) + 2R_X^2(\tau)$. 
Problem 4

(a) For the sign or signum function defined as

\[
\text{sgn}(t) = \begin{cases} 
1, & t > 0 \\
-1, & t < 0 \end{cases}
\]

the corresponding Fourier transform is \( \mathcal{F}\{\text{sgn}(t)\} = \frac{2}{\pi} \). By the duality property of the Fourier transform, \( \mathcal{F}\left\{\frac{1}{\pi} \right\} = -\text{sgn}(\omega) = H(\omega) \), which completes the proof.

(b) Let \( \mathcal{F}\{x(t)\} = X(\omega) \). Then, \( \mathcal{F}\{\hat{x}(t)\} = H(\omega) \hat{X}(\omega) = -\text{sgn}(-\omega)X(\omega) \) and therefore, \( \mathcal{F}\{\hat{x}(t)\} = H(\omega)\mathcal{F}\{\hat{x}(t)\} \) = \( H^2(\omega)X(\omega) = -X(\omega) \). Taking the inverse Fourier transform to both sides of the last equation we conclude that \( \hat{x}(t) = -x(t) \).

(c) It is easy to show that (also asked in HW2)

\[
\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\hat{X}(-\omega)d\omega.
\]

Because \( x(t) \) is real, its Fourier transform exhibits conjugate symmetry and since \( \hat{X}(\omega) = -\text{sgn}(\omega)X(\omega) \) we have that \( \hat{X}(-\omega) = -\text{sgn}(-\omega)X(-\omega) = \text{sgn}(\omega)X^*(\omega) \). Therefore,

\[
\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = \frac{2}{2\pi} \int_{-\infty}^{\infty} \text{sgn}(\omega)|X(\omega)|^2d\omega = 0,
\]

since \( \text{sgn}(-\omega)|X(-\omega)|^2 = -\text{sgn}(\omega)|X^*(\omega)|^2 = -\text{sgn}(\omega)|X(\omega)|^2 \), i.e., the integrand is an odd function of \( \omega \).

Problem 5

(a) \( m_X(t) = E\left[ \sum_{n=-\infty}^{\infty} C_n s(t - n\bar{T}) \right] = \sum_{n=-\infty}^{\infty} E[C_n]s(t - n\bar{T}) = \mu \sum_{n=-\infty}^{\infty} s(t - n\bar{T}). \) Clearly, \( m_X(t + \bar{T}) = m_X(t), \forall t \).

(b)

\[
R_X(t + \tau, t) = E[X(t + \tau)X(t)]
\]

\[
= E\left[ \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} C_n C_m s(t + \tau - n\bar{T})s(t - m\bar{T}) \right]
\]

\[
= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[C_n C_m] s(t + \tau - n\bar{T})s(t - m\bar{T})
\]

\[
= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R_C(n - m)s(t + \tau - n\bar{T})s(t - m\bar{T})
\]
HW 1: Solution

1-3

By inspection we can see that \( \psi \) the functions \( v \) and equality on the left if and only if

\[
\| v \| = \| w \| \iff \psi_v = \psi_w
\]

In addition, by \( v \parallel w \) if \( w \parallel v \), we conclude that \( \psi_v = \psi_w \). The Pythagorean theorem implies then that \( \| v - w \| = \| v - w^\perp \| + \| w^\perp \| \) and since \( \| v - w \| \geq 0 \) we conclude that \( \| v^\perp \| = \| v - v \| \leq \| v - w \| \), \( \forall w \in \mathcal{W} \). The equality holds if and only if \( \| v - w \| = 0 \), i.e., \( w = v \).

Problem 6

(a) To show that the waveforms are orthogonal, we need to verify that

\[
\int_{-\infty}^{\infty} \psi_m(t)\psi_n(t)dt = 0, \ \forall m \neq n.
\]

These integrations are straightforward and the orthogonality can be readily verified. We can also verify the orthogonality by inspection by noting that \( \psi_1(t) \) is constant over \([0, 4]\) while \( f_2^3 \psi_2(t)dt = f_2^3 \psi_3(t)dt = 0 \), which make the inner product zero for \((m, n) = (1, 2)\) and \((m, n) = (1, 3)\). Moreover, \( \psi_2(t) \) is a constant over \([0, 2]\) and \([2, 4]\), while \( f_0^2 \psi_3(t)dt = f_2^4 \psi_3(t)dt = 0 \), which make the inner product zero for \((m, n) = (2, 3)\).

Finally, \( f_0^4 \psi_1^2(t)dt = f_0^4 \psi_2^2(t)dt = f_0^4 \psi_3^2(t)dt = 1 \), hence we have orthonormality.

(b) To determine the linear combination \( x(t) = c_1 \psi_1(t) + c_2 \psi_2(t) + c_3 \psi_3(t) \), we need to determine the corresponding coefficients

\[
c_k = \int_{0}^{4} x(t)\psi_k(t)dt, \ \ k = 1, 2, 3.
\]

Performing these straightforward integrations we obtain \( c_1 = c_2 = c_3 = 0 \) and therefore, \( x(t) \perp \text{span}\{\psi_1, \psi_2, \psi_3\} \).

Problem 7

By inspection we can see that \( s_3(t) = s_2(t) - s_1(t) \). Therefore, \( \dim (\text{span}\{s_1, s_2, s_3, s_4\}) \leq 3 \). A set of orthonormal functions can be found using the Gram-Schmidt procedure via straightforward integrations, by considering the given waveforms in any order of preference. Alternatively, we can determine an orthonormal basis by inspection. Consider the functions \( \psi_1(t) = 1, 0 \leq t \leq 1, \psi_2(t) = 1, 1 \leq t \leq 2 \) and \( \psi_3(t) = 1, 2 \leq t \leq 3 \) assuming that they take the value zero for any other \( t \). Clearly, this set is orthonormal and also \( \text{span}\{s_1, s_2, s_3, s_4\} = \text{span}\{\psi_1, \psi_2, \psi_3\} \).

Problem 8

We first note that \( v = v_W + v_W^\perp \), where \( v_W \perp v_W^\perp \). By the Pythagorean theorem, \( \| v \|^2 = \| v_W \|^2 + \| v_W^\perp \|^2 \) and since \( \| v_W^\perp \| \geq 0 \) it follows that \( 0 \leq \| v_W \| \leq \| v \| \) with equality on the right if and only if \( \| v_W^\perp \| = 0 \) or \( v_W^\perp = 0 \), i.e., \( v \in \mathcal{W} \) and equality on the left if and only if \( v = v_W^\perp \), i.e., \( v \in \mathcal{W}^\perp \).

In addition, by \( v = v_W + v_W^\perp \) and for any \( w \in \mathcal{W} \) we have that \( v - w = (v_W - w) + v_W^\perp \). Since \( w \in \mathcal{W} \), we obtain that \( v_W - w \in \mathcal{W} \) and therefore, \( v_W - w \perp v_W^\perp \). The Pythagorean theorem implies then that

\[
\| v - w \| = \| v_W - w \|^2 + \| v_W^\perp \|^2
\]

and since \( \| v_W - w \| \geq 0 \) we conclude that \( \| v_W^\perp \| = \| v - w \| \leq \| v - w \|, \forall w \in \mathcal{W} \). The equality holds if and only if \( \| v_W - w \| = 0 \), i.e., \( w = v_W \).