ECE 562 Advanced Digital Communications **University of Illinois at Urbana-Champaign**
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HW1 Solution

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Problem 1

Since $X \sim \mathcal{N}(0, \sigma^2 I_n), Y \sim \mathcal{N}(m_Y, \Sigma_Y)$ with $m_Y = E[AX] = AE[X] = 0$ and $\Sigma_Y = E[(Y - m_Y)(Y - m_Y)^T] = A\Sigma_X A^T = \sigma^2 AA^T$. Clearly, X, Y are identically distributed when $\Sigma_Y = \Sigma_X$, i.e., $AA^T = I_n$. In other words, X, Y are identically distributed when A is an orthogonal matrix.

Problem 2

By the hint, $E[X_1X_2X_3X_4] = (-\jmath)^4 \left.\frac{\partial^4 \phi_X(u_1,u_2,u_3,u_4)}{\partial u_1 \partial u_2 \partial u_3 \partial u_4}\right|_{u=0} = \left.\frac{\partial^4 \phi_X(u_1,u_2,u_3,u_4)}{\partial u_1 \partial u_2 \partial u_3 \partial u_4}\right|_{u=0}$. Moreover, for a (degenerate or non-degenerate) Guassian random vector X,

$$\phi_X(u) = E\left[e^{\jmath u^T X}\right] = e^{\jmath u^T m_X - \frac{1}{2} u^T \Sigma_X u}$$

and since $X \sim \mathcal{N}(0, \Sigma)$ we have that $\phi_X(u) = e^{-\frac{1}{2} u^T \Sigma u} = e^{-\frac{1}{2}\sum_{i,j=1}^4 u_i \Sigma_{ij} u_j}$. Clearly,

$$\begin{aligned}
\frac{\partial \phi_X(u)}{\partial u_k} &= e^{-\frac{1}{2}\sum_{i,j=1}^4 u_i \Sigma_{ij} u_j} \left[-\frac{1}{2}\left(\sum_{j=1}^4 \Sigma_{kj} u_j + \sum_{i=1}^4 u_i \Sigma_{ik}\right)\right] \\
&= e^{-\frac{1}{2}\sum_{i,j=1}^4 u_i \Sigma_{ij} u_j} \left[-\frac{1}{2}\left(\sum_{j=1}^4 \Sigma_{kj} u_j + \sum_{i=1}^4 u_i \Sigma_{ki}\right)\right] \\
&= -\sigma_k^T u e^{-\frac{1}{2}\sum_{i,j=1}^4 u_i \Sigma_{ij} u_j}. \qquad (1.1)
\end{aligned}$$

Here, $\Sigma_{ik} = \Sigma_{ki}$ has been used and σ_k^T denotes the kth row of Σ. Furthermore,

$$\frac{\partial \sigma_k^T u}{\partial u_m} = \frac{\partial}{\partial u_m}\left(\sum_{j=1}^4 \Sigma_{kj} u_j\right) = \Sigma_{km} = \Sigma_{mk}. \qquad (1.2)$$

Performing the complete differentiation for $E[X_1X_2X_3X_4]$ by employing (1.1) and (1.2) the result follows.

Problem 3

By the definition of the autocorrelation function $R_Y(t+\tau, t) = E[Y(t+\tau)Y(t)] = E[X^2(t+\tau)X^2(t)]$. By employing the property in Problem 2 for $X_1 = X(t+\tau), X_2 = X(t+\tau), X_3 = X(t), X_4 = X(t)$ we obtain

$$R_Y(t+\tau, t) = R_X(t+\tau, t+\tau)R_X(t,t) + 2R_X^2(t+\tau, t).$$

For a mean-zero stationary Gaussian process, this equation takes the form $R_Y(\tau) = R_X^2(0) + 2R_X^2(\tau)$.

Problem 4

(a) For the *sign* or *signum* function defined as

$$\mathrm{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

the corresponding Fourier transform is $\mathcal{F}\{\mathrm{sgn}(t)\} = \frac{2}{\jmath\omega}$. By the *duality property* of the Fourier transform, $\mathcal{F}\left\{\frac{2}{\jmath t}\right\} = 2\pi\mathrm{sgn}(-\omega) = -2\pi\mathrm{sgn}(\omega)$. Multiplying both sides of the last equation with $\frac{\jmath}{2\pi}$ we obtain that $\mathcal{F}\left\{\frac{1}{\pi t}\right\} = -\jmath\mathrm{sgn}(\omega) = H(\omega)$, which completes the proof.

(b) Let $\mathcal{F}\{x(t)\} = X(\omega)$. Then, $\mathcal{F}\{\hat{x}(t)\} = H(\omega)X(\omega) = -\jmath\mathrm{sgn}(\omega)X(\omega)$ and therefore, $\mathcal{F}\left\{\hat{\hat{x}}(t)\right\} = H(\omega)\mathcal{F}\{\hat{x}(t)\} = H^2(\omega)X(\omega) = -X(\omega)$. Taking the inverse Fourier transform to both sides of the last equation we conclude that $\hat{\hat{x}}(t) = -x(t)$.

(c) It is easy to show that (also asked in HW2)

$$\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\hat{X}(-\omega)d\omega.$$

Because $x(t)$ is real, its Fourier transform exhibits conjugate symmetry and since $\hat{X}(\omega) = -\jmath\mathrm{sgn}(\omega)X(\omega)$ we have that $\hat{X}(-\omega) = -\jmath\mathrm{sgn}(-\omega)X(-\omega) = \jmath\mathrm{sgn}(\omega)X^*(\omega)$. Therefore,

$$\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = \frac{\jmath}{2\pi}\int_{-\infty}^{\infty} \mathrm{sgn}(\omega)|X(\omega)|^2 d\omega = 0,$$

since $\mathrm{sgn}(-\omega)|X(-\omega)|^2 = -\mathrm{sgn}(\omega)|X^*(\omega)|^2 = -\mathrm{sgn}(\omega)|X(\omega)|^2$, i.e., the integrand is an odd function of ω.

Problem 5

(a) $m_X(t) = E\left[\sum_{n=-\infty}^{\infty} C_n s(t - n\tilde{T})\right] = \sum_{n=-\infty}^{\infty} E[C_n]s(t - n\tilde{T}) = \mu\sum_{n=-\infty}^{\infty} s(t - n\tilde{T})$. Clearly, $m_X(t + \tilde{T}) = m_X(t), \forall t$.

(b)

$$\begin{aligned} R_X(t+\tau, t) &= E[X(t+\tau)X(t)] \\ &= E\left[\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} C_n C_m s(t+\tau-n\tilde{T})s(t-m\tilde{T})\right] \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E\left[C_n C_m\right] s(t+\tau-n\tilde{T})s(t-m\tilde{T}) \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_C(n-m)s(t+\tau-n\tilde{T})s(t-m\tilde{T}) \end{aligned}$$

(b) $m_X(t)$ is $\tilde{T}$-periodic by part (a). Moreover,

$$\begin{aligned} R_X(t+\tau+\tilde{T}, t+\tilde{T}) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R_C(n-m) s(t+\tilde{T}+\tau-n\tilde{T}) s(t+\tilde{T}-m\tilde{T}) \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R_C(n-m) s(t+\tau-(n-1)\tilde{T}) s(t-(m-1)\tilde{T}) \\ &= \sum_{n'=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} R_C(n'-m') s(t+\tau-n'\tilde{T}) s(t-m'\tilde{T}) = R_X(t+\tau, t), \end{aligned}$$

where the changes of variables $n' = n-1, m' = m-1$ have been performed. Thus, the process is (wide-sense) cyclostationary.

Problem 6

(a) To show that the waveforms are orthogonal, we need to verify that

$$\int_{-\infty}^{\infty} \psi_m(t)\psi_n(t)dt = 0, \ \ \forall m \neq n.$$

These integrations are straightforward and the orthogonality can be readily verified. We can also verify the orthogonality by inspection by noting that $\psi_1(t)$ is constant over $[0,4]$ while $\int_0^4 \psi_2(t)dt = \int_0^4 \psi_3(t)dt = 0$, which make the inner product zero for $(m,n) = (1,2)$ and $(m,n) = (1,3)$. Moreover, $\psi_2(t)$ is a constant over $[0,2]$ and $[2,4]$, while $\int_0^2 \psi_3(t)dt = \int_2^4 \psi_3(t)dt = 0$, which make the inner product zero for $(m,n) = (2,3)$.

Finally, $\int_0^4 \psi_1^2(t)dt = \int_0^4 \psi_2^2(t)dt = \int_0^4 \psi_3^2(t)dt = 1$, hence we have orthonormality.

(b) To determine the linear combination $x(t) = c_1\psi_1(t) + c_2\psi_2(t) + c_3\psi_3(t)$, we need to determine the corresponding coefficients

$$c_k = \int_0^4 x(t)\psi_k(t)dt, \quad k = 1,2,3.$$

Performing these straightforward integrations we obtain $c_1 = c_2 = c_3 = 0$ and therefore, $x(t) \perp \text{span}\{\psi_1, \psi_2, \psi_3\}$.

Problem 7

By inspection we can see that $s_3(t) = s_2(t) - s_1(t)$. Therefore, $\dim(\text{span}\{s_1, s_2, s_3, s_4\}) \leq 3$. A set of orthonormal functions can be found using the Gram-Schmidt procedure via straightforward integrations, by considering the given waveforms in any order of preference. Alternatively, we can determine an orthonormal basis by inspection. Consider the functions $\psi_1(t) = 1, 0 \leq t \leq 1$, $\psi_2(t) = 1, 1 \leq t \leq 2$ and $\psi_3(t) = 1, 2 \leq t \leq 3$ assuming that they take the value zero for any other t. Clearly, this set is orthonormal and also $\text{span}\{s_1, s_2, s_3, s_4\} = \text{span}\{\psi_1, \psi_2, \psi_3\}$.

Problem 8

We first note that $v = v_{\mathcal{W}} + v_{\mathcal{W}}^{\perp}$, where $v_{\mathcal{W}} \perp v_{\mathcal{W}}^{\perp}$. By the Pythagorean theorem, $\|v\|^2 = \|v_{\mathcal{W}}\|^2 + \|v_{\mathcal{W}}^{\perp}\|^2$ and since $\|v_{\mathcal{W}}^{\perp}\| \geq 0$ it follows that $0 \leq \|v_{\mathcal{W}}\| \leq \|v\|$ with equality on the right if and only if $\|v_{\mathcal{W}}^{\perp}\| = 0$ or $v_{\mathcal{W}}^{\perp} = 0$, i.e., $v \in \mathcal{W}$ and equality on the left if and only if $v = v_{\mathcal{W}}^{\perp}$, i.e., $v \in \mathcal{W}^{\perp}$.

In addition, by $v = v_{\mathcal{W}} + v_{\mathcal{W}}^{\perp}$ and for any $w \in \mathcal{W}$ we have that $v - w = (v_{\mathcal{W}} - w) + v_{\mathcal{W}}^{\perp}$. Since $w \in \mathcal{W}$, we obtain that $v_{\mathcal{W}} - w \in \mathcal{W}$ and therefore, $v_{\mathcal{W}} - w \perp v_{\mathcal{W}}^{\perp}$. The Pythagorean theorem implies then that $\|v-w\|^2 = \|v_{\mathcal{W}} - w\|^2 + \|v_{\mathcal{W}}^{\perp}\|^2$ and since $\|v_{\mathcal{W}} - w\| \geq 0$ we conclude that $\|v_{\mathcal{W}}^{\perp}\| = \|v - v_{\mathcal{W}}\| \leq \|v - w\|, \forall w \in \mathcal{W}$. The equality holds if and only if $\|v_{\mathcal{W}} - w\| = 0$, i.e., $w = v_{\mathcal{W}}$.