ECE534 Random Processes **University of Illinois at Urbana-Champaign**
Spring 2020

Midterm 2: Solution

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Problem 1

(a) Since (Z,Y) are independent Gaussian random variables, they are jointly Gaussian. Moreover,

$$\begin{bmatrix} Z+Y \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ Y \end{bmatrix}.$$

Therefore, $(Z+Y,Y)$ are jointly Gaussian. Since $f_{X,Y} = f_{Z+Y,Y}$, we conclude that (X,Y) are jointly Gaussian.

(b) Since (X,Y) are jointly Gaussian, $E[Y|X]$ is linear and it is given by

$$E[Y|X] = E[Y] + \frac{\text{Cov}(X,Y)}{\text{Var}(X)}(X - E[X]).$$

By $f_{X,Y} = f_{Z+Y,Y}$ we have that $E[X] = E[Z+Y] = \mu$, $\text{Var}(X) = \text{Var}(Z+Y) = 1+\sigma^2$ and

$$\text{Cov}(X,Y) = \text{Cov}(Z+Y,Y) = \text{Cov}(Z,Y) + \text{Cov}(Y,Y) = \sigma^2.$$

By combining

$$E[Y|X] = \mu + \frac{\sigma^2}{1+\sigma^2}(X-\mu).$$

Problem 2

By Markov's inequality

$$P(|X_n - X| > \epsilon) = P(|X_n - X|^2 > \epsilon^2) \leq \frac{E[|X_n - X|^2]}{\epsilon^2}.$$

Therefore,

$$\sum_{n=1}^{\infty} P(|X_n - X| > \epsilon) \leq \sum_{n=1}^{\infty} \frac{E[|X_n - X|^2]}{\epsilon^2} < \infty.$$

By part (a) in the Borel-Cantelli lemma we conclude that $X_n \xrightarrow[n\to\infty]{a.s} X$.

Problem 3

Let $C = \left\{|X| \geq \frac{c}{4}\right\}$. Then,

$$\begin{aligned} \frac{1}{4} \leq E[|X|] &= E[|X|\mathbb{1}_C] + E[|X|\mathbb{1}_C^c] \\ &\leq \sqrt{E[X^2]E[\mathbb{1}_C^2]} + \frac{c}{4} \\ &= \sqrt{P(C)} + \frac{c}{4}. \end{aligned}$$

Therefore,

$$P\left(|X| \geq \frac{c}{4}\right) \geq \frac{(1-c)^2}{16}.$$

Problem 4

We first note that $Y_n > 0, \forall n \geq 1$ almost surely since f, g are densities. Therefore,

$$\begin{aligned} E[|Y_n|] = E[Y_n] = E\left[\prod_{k=1}^{n} \frac{g(X_k)}{f(X_k)}\right] \\ = \int \prod_{k=1}^{n} g(x_k) dx_1 \cdots dx_n = 1 < \infty, \ \ \forall n \geq 1, \end{aligned}$$

due to the i.i.d. assumption on $(X_k)_{k \geq 1}$ and the fact that f, g are densities.

Moreover,

$$\begin{aligned} E[Y_{n+1}|X_n, \ldots, X_1] = E\left[\prod_{k=1}^{n+1} \frac{g(X_k)}{f(X_k)}|X_n, \ldots, X_1\right] \\ = \prod_{k=1}^{n} \frac{g(X_k)}{f(X_k)} E\left[\frac{g(X_{n+1})}{f(X_{n+1})}\right] = Y_n \end{aligned}$$

by using the observation that $E\left[\frac{g(X_{n+1})}{f(X_{n+1})}\right] = \int g(x_{n+1}) dx_{n+1} = 1.$

Problem 5

(a) The chain is not irreducible, since, e.g., states $2, 3$ are not reachable by state 1. Moreover, it is easy to see that all states have period 1. Therefore, the chain is aperiodic.

(b) Solving the balance equations

$$\pi = \pi P$$

by taking into account the constraint $\pi_1 + \pi_2 + \pi_3 = 1$, we obtain

$$\pi_1 = \pi_1, \ \ \pi_2 = \pi_2, \ \ \pi_3 = 0$$

subject to $\pi_1 + \pi_2 + \pi_3 = 1$. Therefore, the chain has infinite many stationary distributions, which are elements of the set

$$\Pi = \{x \in [0, 1] : \pi = [x \ \ 1 - x \ \ 0]\}.$$

(c) Clearly, $\lim_{n \to \infty} p_k = [1/2 \ \ 1/2 \ \ 0]$, since $p_0 \in \Pi$.