ECE534, Spring 2020: Midterm #2

Problem 1 (25 pts)

Suppose that the conditional distribution of X given $Y = y$ is $\mathcal{N}(y, 1)$, i.e., Gaussian with mean y and variance 1. Denote by $f_{X|Y}(x|y)$ the corresponding density function. Moreover, let $Y \sim \mathcal{N}(\mu, \sigma^2)$ and consider the random variable $Z \sim \mathcal{N}(0, 1)$, which is independent of Y. Based on these assumptions, it is easy to show that $f_{X,Y} = f_{Z+Y,Y}$, i.e., the joint densities of (X, Y) and $(Z + Y, Y)$ are the same.

(a) Show that (X, Y) are jointly Gaussian.

(b) Find $E[Y|X]$.

Problem 2 (20 pts)

Prove that $X_n \xrightarrow[n\to\infty]{a.s} X$ whenever $\sum_{n=1}^{\infty} E[|X_n - X|^2] < \infty$.
Hint: Use the Borel-Cantelli lemma.

Problem 3 (15 pts)

Let X be a random variable such that $E[|X|] \geq 1/4$ and $E[X^2] = 1$. Show that for any $c \in [0, 1]$,

$$P\left(|X| \geq \frac{c}{4}\right) \geq \frac{(1-c)^2}{16}.$$

Hint: Start by choosing an appropriate event C and express $E[|X|] = E[|X|\mathbb{1}_C] + E[|X|\mathbb{1}_{C^c}]$.

Problem 4 (25 pts)

Let f, g be two *equivalent* densities for all x, i.e., $f(x) = 0$ if and only if $g(x) = 0$. Consider a sequence $(X_k)_{k\geq1}$ of i.i.d. random variables drawn from f. Let $Y_n = \prod_{k=1}^{n} \frac{g(X_k)}{f(X_k)}$, where $\prod_{k=1}^{n} a_k$ denotes the product of $a_1, a_2, \ldots, a_n$. Show that $(Y_k)_{k\geq1}$ is a martingale with respect to $(X_k)_{k\geq1}$.

Problem 5 (15 pts)

Consider a Markov chain with state space $\{1, 2, 3\}$ and transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

(a) Is this chain irreducible? Aperiodic? Justify your answers.

(b) Is the stationary distribution of this chain unique? If yes, then compute it, otherwise determine all possible stationary distributions of this chain.

(c) Assuming that the initial distribution is $p_0 = [P(X_0 = 1) \quad P(X_0 = 2) \quad P(X_0 = 3)] = [1/2 \quad 1/2 \quad 0]$, determine $\lim_{k\to\infty} p_k$, where $p_k = [P(X_k = 1) \quad P(X_k = 2) \quad P(X_k = 3)]$.