# ECE534, Spring 2020: Midterm #1

**Problem 1**

Consider a coin with $P(H) = p$. The coin is tossed repeatedly. Let $p_n$ denote the probability that in $n$ tosses an even number of $H$ appears, with 0 being an even number. Similarly to HW2, find a recursion for $p_n$ and identify the appropriate boundary condition $p_0$.

**Solution**

Clearly, the recursion is

$$p_n = p(1 - p_{n-1}) + (1 - p)p_{n-1}, \quad n \geq 1$$

and $p_0 = 1$.

**Problem 2**

Let $X \sim \text{Pois}(\lambda)$. Use LOTUS to show that

$$E[X^n] = \lambda E[(X + 1)^{n-1}].$$

**Solution**

By LOTUS:

$$\begin{aligned} E[X^n] &= \sum_{k=0}^{\infty} k^n e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k^n e^{-\lambda} \frac{\lambda^k}{k!} \\ &= \lambda \sum_{k=1}^{\infty} k^{n-1} e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!} \\ &= \lambda \sum_{m=0}^{\infty} (1 + m)^{n-1} e^{-\lambda} \frac{\lambda^m}{m!} \\ &= \lambda E[(X + 1)^{n-1}]. \end{aligned}$$

**Problem 3**

Suppose that $X \sim \mathcal{N}(0,1)$. Recall the proof of the Gaussian tail bound derived in class, namely

$$P(X \geq x) \leq \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \leq \frac{1}{x}e^{-\frac{x^2}{2}}, \quad \forall x > 0.$$

The purpose of this problem is to give an alternative proof of the same result. Start by expressing

$$P(X \geq x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,dt.$$

Then, introduce a change of variables to make the lower limit of the integral equal to 0 and immediately finish the proof by appropriately upper bounding the resulting integral.

**Solution**

$$P(X \geq x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,dt.$$

Introduce the change of variables $u = t - x$ to obtain

$$\begin{aligned} P(X \geq x) &= \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{(u+x)^2}{2}}du = \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}e^{-\frac{x^2}{2}}e^{-ux}du \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\int_0^\infty e^{-\frac{u^2}{2}}e^{-ux}du \leq \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\int_0^\infty e^{-ux}du \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\left[-\frac{e^{-ux}}{x}\right]_0^\infty = \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}. \end{aligned}$$

In the first inequality, the fact that $e^{-\frac{u^2}{2}} \leq 1, \forall u \in \mathbb{R}$ has been used.
**Alternative Solution**: As before,

$$\begin{aligned} P(X \geq x) &= \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{(u+x)^2}{2}}du = \int_0^\infty \frac{u+x}{u+x}\frac{1}{\sqrt{2\pi}}e^{-\frac{(u+x)^2}{2}}du \\ &\leq \frac{1}{\sqrt{2\pi}x}\int_0^\infty \left[-e^{-\frac{(u+x)^2}{2}}\right]' du = \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}. \end{aligned}$$

**Problem 4**

1. Let $X$ be a random variable. Assume that $E\left[e^{X^2}\right] \leq 2$. Show that

$$P(|X| > t) \leq 2e^{-t^2}, \quad \forall t \geq 0.$$

**Solution**

Using Markov's inequality and the Chernoff trick with $\lambda = 1$

$$\begin{aligned} P(|X| > t) = P(X^2 > t^2) &= P\left(e^{X^2} \geq e^{t^2}\right) \\ &\leq E\left[e^{X^2}\right] e^{-t^2} \leq 2e^{-t^2}, \quad \forall t > 0. \end{aligned}$$

2. Let $Z$ be a random variable. Suppose that $E[Z] = 0$ and $E[|Z|^p] \leq p^p, \forall p \geq 1$. Show that

$$E\left[e^{\lambda Z}\right] \leq e^{2e^2\lambda^2}, \quad \forall \lambda \text{ such that } |\lambda| \leq \frac{1}{2e}$$

starting your derivation as

$$E\left[e^{\lambda Z}\right] = E\left[1 + \lambda Z + \sum_{p=2}^{\infty} \frac{(\lambda Z)^p}{p!}\right] = \cdots$$

**Note**: $E[Z^p] \leq E[|Z|^p] \leq p^p, \forall p \geq 1$. Also, $1 + x \leq e^x, \forall x \in \mathbb{R}$ and $(\frac{p}{e})^p \leq p!$.

**Solution**

$$\begin{aligned} E\left[e^{\lambda Z}\right] &= E\left[1 + \lambda Z + \sum_{p=2}^{\infty} \frac{(\lambda Z)^p}{p!}\right] = \left|E\left[1 + \lambda Z + \sum_{p=2}^{\infty} \frac{(\lambda Z)^p}{p!}\right]\right| \leq 1 + \sum_{p=2}^{\infty} \frac{|\lambda|^p E[|Z|^p]}{p!} \\ &\leq 1 + \sum_{p=2}^{\infty} \frac{|\lambda|^p p^p}{(\frac{p}{e})^p} = 1 + \sum_{p=2}^{\infty} (e|\lambda|)^p = 1 + (e|\lambda|)^2 \sum_{p=0}^{\infty} (e|\lambda|)^p \\ &= 1 + \frac{(e\lambda)^2}{1 - e|\lambda|} \leq 1 + 2(e\lambda)^2 \leq e^{2e^2\lambda^2}. \end{aligned}$$