ECE534 Random Processes **University of Illinois at Urbana-Champaign**

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Homework 5

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Problem 1. (Poisson process). Let N_t be a Poisson process with rate λ, and assume that $X_1, X_2, \ldots X_n$ are the corresponding inter-arrival times.

(a) Find the covariance function $C_N(t_1, t_2)$ of N_t for $t_1, t_2 \in [0, \infty)$.

(b) Suppose that one starts observing the process at time $t = S$. Given that the last arrival happened at time $t = S - 1$, find the probability that the next arrival happens before time $t = S + 1$.

(c) Assume that only one arrival happens up to time $t = T$, i.e. $N_T = 1$. Find the cumulative distribution function of the first arrival time X_1, i.e. $P[X_1 \leq x | N_T = 1]$.

(d) Given that k arrivals occur in the interval $[0, T]$, i.e. $N_T = k$, find the probability that the k^{th} arrival doesn't happen before time $t = \alpha T$ for some $\alpha < 1$.

(**Hint**: Given $N_T = k$, the arrival times of the k events are order statistics of i.i.d. uniform random variables in the interval $[0, T]$, i.e. they are uniform random variables sorted in ascending order.)

Problem 2. (Branching process). Consider a branching process with offspring distribution given by $p_0 = \frac{1}{3}$, $p_1 = \frac{1}{2}$ and $p_2 = \frac{1}{6}$. Consider a population beginning with one individual, comprising generation zero. Compute the following quantities:

(a) Expected number of individuals at generation k.

(b) Expected number of individuals that have no offspring.

(c) Probability of extinction by generation 3 (but not by generation 2).

(d) Identify a sufficient condition on the offspring distribution p such that a branching process dies out eventually. Does the given branching process die out?

Problem 3. (Kalman Filtering). Consider the state and observation equations given by

$$x_{k+1} = 2x_k + w_k$$
$$y_k = x_k + v_k$$

where $w_1, w_2, \ldots, v_1, v_2, \ldots$ are independent $\mathcal{N}(0, 1)$ random variables and $x_0 = 0$.

(a) First, consider estimating x_k using the LMMSE estimator of x_k given y_k alone. Compute $\widehat{E}[x_k|y_k]$ and the variance σ_k^2 of the estimation error $e_k = x_k - \widehat{E}[x_k|y_k]$.

(b) What is the limit of σ_k^2 as $k \to \infty$?

(c) Now use the Kalman filter equations to find $\hat{x}_{k|k}$ as a function of $\hat{x}_{k-1|k-1}$ and the estimation error variance $\sigma^2_{k|k}$ as a function of $\sigma^2_{k-1|k-1}$.

(d) What is the limit of $\sigma^2_{k|k}$ as $k \to \infty$?

(**Hint:** You can assume that the limit exists and set $\sigma^2_{k|k} = \sigma^2_{k-1|k-1} = \sigma^2_\infty$ in (c).)

Problem 4. (WSS processes and LTI systems). A wide-sense stationary Gaussian process $X(t)$ has auto-correlation function $R_x(\tau) = 5e^{-2|\tau|}$. For each of the statements below, state whether it is true or false and explain your answer.

(a) If $X(t)$ is the input to an LTI system with impulse response $h(t) = e^{-2t}u(t)$, then the output process is white. A process is called white if it has constant power spectral density.

(b) If $X(t)$ is the input to an LTI system with impulse response $h(t) = e^{-t^2}$, then the output process and the input process are orthogonal. Two processes are called orthogonal if their cross-correlation is 0.

(**Hint:** Note that the impulse response has a form similar to the Gaussian pdf.)

(c) $Y(t) = X^2(t)$ is a Gaussian process.

Problem 5. (Ergodicity) Let $X(n)$ be a two-sided, strictly stationary and ergodic process. For each of the following processes: (i) find the mean $\mu_Y(n)$ in terms of μ_X, (ii) determine whether the process is strictly stationary, and (iii) determine whether the process is ergodic in the mean. Explain your answers.

(a) $Y(n) = \frac{1}{5}\sum_{k=-2}^{2} X(n-k)$

(b) $Y(n) = \begin{cases} X(n), & \text{if } A > 5 \\ 0, & \text{if } A \leq 5, \end{cases}$ where A is a Poisson random variable with parameter λ. A and $X(n)$ are independent.

Problem 6. (Wiener filtering) Problem 9.13 from Prof. Hajek's book.

(**Hint:** Note that in a block diagram representation, each rectangular block represents a filter with an associated transfer function. If we denote the impulse responses of the first and the second filters as k_1 and k_2, respectively, then $X_{out}(t) = X(t) * k_1(t) * k_2(t)$ and $N_{out}(t) = N(t) * k_2(t)$ where '$*$' denotes the convolution operation.)

Problem 7. (Azuma-Hoeffding inequality) You throw a hundred balls, one at a time, into 20 bins. Each ball can fall in any one of the bins uniformly at random. Let X be the number of bins containing five balls or more after all balls are thrown. Find an upper bound to $P(X \geq 15)$.

(**Hint:** Let Z_k be the index of the bin where the k^{th} ball falls. Show that $M_k = E[X|Z_1, \cdots, Z_k]$ is a martingale with bounded increments.)

Problem 8. (Geometric Brownian motion) Let $W(t)$ be a standard Brownian motion. Define

$$X(t) = \exp\{W(t)\}, \quad \text{for all t } \in [0, \infty).$$

(a) Find $E[X(t)]$ for all $t \in [0, \infty)$.

(b) Find $Var(X(t))$ for all $t \in [0, \infty)$.

(c) Let $0 \leq s \leq t$. Find Cov$(X(s), X(t))$.

(**Hint:** Think about the moment generating function of a Gaussian random variable.)