ECE534 Random Processes | **University of Illinois at Urbana-Champaign**
Spring 2020 | **Gaussianity, Markov Chains and Martingales**

HW 4: Solution

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Problem 1

1. Let $\alpha = \frac{1}{2}\left(\frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}}\right)$ and $\beta = \frac{1}{2}\left(\frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}}\right)$. Note that any linear combination of W, Z corresponds to a linear combination of X, Y:

$$aW + bZ = (a\alpha + b\beta)X + (a\beta + b\alpha)Y,$$

which is Gaussian since X, Y are jointly Gaussian. Therefore, W and Z are jointly Gaussian.

2. From (a), W and Z have a bi-variate Gaussian density, determined by the mean vector and covariance matrix. Clearly, $\mu_W = \mu_Z = 0$. For the variances and the correlation coefficient, we have:

$$\sigma_Z^2 = \sigma_W^2 = \mathrm{Cov}(W, W) = \frac{1-\rho^2}{(1+\rho)(1-\rho)} = 1,$$
$$\rho_{W,Z} = \mathrm{Cov}(W, Z) = \frac{-\rho+\rho}{(1+\rho)(1-\rho)} = 0.$$

Therefore, W Z are uncorrelated (hence independent, being jointly Gaussian). This shows that for $w, z \in \mathbb{R}$:

$$f_{WZ}(w,z) = f_W(w)f_Z(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}} = \frac{1}{2\pi}e^{-\frac{w^2+z^2}{2}}.$$

3. The MMSE estimator of Z given W is the conditional mean $E[Z|W]$. Since Z, W are independent, we have

$$E[Z|W] = E[Z] = \mu_Z = 0.$$

4. A straightforward computation gives $\mathrm{Cov}(X, W) = \frac{1}{2}\left(\sqrt{1+\rho} + \sqrt{1-\rho}\right)$. We now have:

$$\hat{E}[X|W] = \mu_X + \frac{\mathrm{Cov}(X, W)}{\sigma_W^2}(W - \mu_W) = \frac{1}{2}\left(\sqrt{1+\rho} + \sqrt{1-\rho}\right)W.$$

Problem 2

By Bayes rule:

$$P(X_n = 1|Y_n = 1) = \frac{P(Y_n = 1|X_n = 1)P(X_n = 1)}{P(Y_n = 1)}.$$

Also, by the definition of Y_n,

$$P(Y_n = 1|X_n = 1) = P(Y_n = X_n) = 0.9.$$

We now solve for the stationary distribution of the DTMC by solving for $\pi = (\pi_1, \pi_2)$ the equation $\pi P = \pi$ subject to $\pi_1 + \pi_2 = 1$. The solution is:

$$\pi_1 = 1/3, \;\; \pi_2 = 2/3.$$

In this case, one can check that the stationary distribution is also limiting and therefore

$$\lim_{n\to\infty} P(X_n = 1) = \pi_1 = 1/3, \quad \lim_{n\to\infty} P(X_n = 2) = \pi_2 = 2/3.$$

Also, by total probability

$$\begin{aligned}\lim_{n\to\infty} P(Y_n = 1) &= \lim_{n\to\infty} P(Y_n = 1|X_n = 1)P(X_n = 1) + P(Y_n = 1|X_n = 2)P(X_n = 2) \\ &= \frac{1}{3}0.9 + \frac{2}{3}0.1.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} P(X_n = 1|Y_n = 1) = P(Y_n = 1|X_n = 1) \lim_{n\to\infty} \frac{P(X_n = 1)}{P(Y_n = 1)} = \frac{0.9\frac{1}{3}}{\frac{1}{3}0.9 + \frac{2}{3}0.1} = \frac{9}{11}.$$

Problem 3

Let X be a finite state Markov chain. Then,

$$|M_t| \le |f(X_t)| + |f(X_0)| + \sum_{s=0}^{t-1} |(E[f(X_{s+1})|X_s] - f(X_s))| < \infty.$$

Additionally,

$$M_{t+1} - M_t = f(X_{t+1}) - f(X_t) - E[f(X_{t+1})|X_t] + f(X_t) = f(X_{t+1}) - E[f(X_{t+1})|X_t].$$

Therefore,

$$E[M_{t+1} - M_t|X_0, X_1, \ldots, X_t] = E[f(X_{t+1})|X_t] - E[f(X_{t+1})|X_t] = 0,$$

which shows that M_t is a martingale with respect to X.

For the reverse implication: Let $f(X_t) = \mathbb{I}_{X_t=x}$. Then,

$$M_{t+1} - M_t = \mathbb{I}_{X_{t+1}=x} - E[\mathbb{I}_{X_{t+1}=x}|X_t].$$

By assumption, $E[M_{t+1} - M_t|X_0, X_1, \ldots, X_t] = 0$ since M is a martingale with respect to X. By conditioning, we obtain:

$$E[\mathbb{I}_{X_{t+1}=x}|X_0, X_1, \ldots, X_t] = E[E[\mathbb{I}_{X_{t+1}=x}|X_t]|X_0, X_1, \ldots, X_t]$$

or

$$P(X_{t+1}|X_0, X_1, \ldots, X_t) = P(X_{t+1}|X_t),$$

since x is arbitrary, i.e., X is a Markov process.