ECE534, Spring 2020: Problem Set #4

Due Apr 10, 2020

1. **Jointly Gaussian Random Variables and MMSE Estimation**

   Suppose that X, Y are jointly Gaussian random variables with $\mu_X = \mu_Y = 0$ and $\sigma_X = \sigma_Y = 1$. Let their correlation coefficient be ρ with $|\rho| < 1$. Based on (X, Y), we define the following random variables:

$$W = \left(\frac{1}{2\sqrt{1+\rho}} + \frac{1}{2\sqrt{1-\rho}}\right) X + \left(\frac{1}{2\sqrt{1+\rho}} - \frac{1}{2\sqrt{1-\rho}}\right) Y$$
$$Z = \left(\frac{1}{2\sqrt{1+\rho}} - \frac{1}{2\sqrt{1-\rho}}\right) X + \left(\frac{1}{2\sqrt{1+\rho}} + \frac{1}{2\sqrt{1-\rho}}\right) Y$$

   (a) Are W, Z jointly Gaussian? Justify your answer.

   (b) Calculate $f_{WZ}(w, z)$.

   (c) Find the MMSE estimator of Z given W.

   (d) Find the linear MMSE estimator of X given W.

2. **DTMCs: Stationary and Limiting Distributions**

   Let $\{X_n\}$ be a Discrete-Time Markov Chain (DTMC) taking values in $\{1, 2\}$ with probability transition matrix

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix},$$

   where $P_{ij} = P(X_{n+1} = j | X_n = i)$. Let $\{Y_n\}$ be a different random process defined as:

$$Y_n = \begin{cases} X_n, & \text{with probability } 0.9 \\ X_n - 1, & \text{with probability } 0.1 \end{cases}.$$

   Find $\lim_{n\to\infty} P(X_n = 1 | Y_n = 1)$.

3. **Markov Chains and Martingales**

   Let X be a random process taking values in a discrete, finite set S. Prove that the following statements are equivalent:

   (a) X is a Markov chain.

   (b) For any function f,

$$M_t = f(X_t) - f(X_0) - \sum_{s=0}^{t-1} \left(E[f(X_{s+1})|X_s] - f(X_s)\right)$$

   is a martingale with respect to X.