HW 3: Solution

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Problem 1

The sequence has only two outcomes (or sample paths) depending on X_1:
$X_1 = 1$: Then $X_1 X_2 \ldots = 10101010\ldots$
$X_1 = 0$: Then $X_1 X_2 \ldots = 01010101\ldots$

Due to the oscillatory nature of the sample paths, it is easy to see that $\{X_n\}$ does not converge almost surely and in probability. Moreover, by Proposition 2.7 in the book, $\{X_n\}$ does not converge in m.s. sense. Finally, as $n \to \infty$, the value of X_n is determined by X_1 and therefore, $X_n \sim \text{Ber}\left(\frac{1}{2}\right), \forall n \geq 1$ (due to X_1), hence $X_n \xrightarrow{d} \text{Ber}\left(\frac{1}{2}\right)$.

Problem 2

Without loss of generality, take $X = 0$. We want to show that $X_n \xrightarrow{p} 0$ if and only if $\lim_{n\to\infty} E\left[\frac{|X_n|}{1+|X_n|}\right] = 0$.

(i) $X_n \xrightarrow{p} 0 \Longrightarrow \lim_{n\to\infty} E\left[\frac{|X_n|}{1+|X_n|}\right] = 0$.

By $X_n \xrightarrow{p} 0$, we have that $\forall \epsilon > 0 : \lim_{n\to\infty} P(|X_n| > \epsilon) = 0$. Note that

$$\frac{|X_n|}{1+|X_n|} \leq \frac{|X_n|}{1+|X_n|}\mathbb{1}[|X_n| > \epsilon] + \epsilon\mathbb{1}[|X_n| \leq \epsilon] \leq \mathbb{1}[|X_n| > \epsilon] + \epsilon.$$

Therefore,

$$E\left[\frac{|X_n|}{1+|X_n|}\right] \leq E[\mathbb{1}[|X_n| > \epsilon]] + \epsilon = P(|X_n| > \epsilon) + \epsilon.$$

Taking the limit, we obtain $\lim_{n\to\infty} E\left[\frac{|X_n|}{1+|X_n|}\right] \leq \epsilon$, and since $\epsilon > 0$ is arbitrary, we have that $\lim_{n\to\infty} E\left[\frac{|X_n|}{1+|X_n|}\right] = 0$.

(ii) $\lim_{n\to\infty} E\left[\frac{|X_n|}{1+|X_n|}\right] = 0 \Longrightarrow X_n \xrightarrow{p} 0$.

Observe that the function $f(x) = \frac{x}{x+1}$ is increasing. Therefore,

$$\frac{\epsilon}{1+\epsilon}\mathbb{1}[|X_n| > \epsilon] \leq \frac{|X_n|}{1+|X_n|}\mathbb{1}[|X_n| > \epsilon] \leq \frac{|X_n|}{1+|X_n|}.$$

Taking expectations and then limits to both sides, we obtain:

$$\frac{\epsilon}{1+\epsilon}\lim_{n\to\infty} P(|X_n|>\epsilon) \le \lim_{n\to\infty} E\left[\frac{|X_n|}{|X_n|+1}\right] = 0$$

Since this holds for any $\epsilon > 0$, we have that $\lim_{n\to\infty} P(|X_n|>\epsilon) = 0,\ \forall\epsilon > 0$. Therefore, $X_n \xrightarrow{p} 0$.

Problem 3

Let $Y_n = \max\{X_1, ..., X_n\}$. For $\epsilon > 0$, we have

$$\begin{aligned}
P\left(\frac{\max(X_1,...,X_n)}{\log n} < 1-\epsilon\right) &= P\left(\frac{Y_n}{\log n} < 1-\epsilon\right)\\
&= P\left(Y_n < (1-\epsilon)\log n\right) = P\left(\cap_{i=1}^n \{X_i < (1-\epsilon)\log n\}\right)\\
&= \left(1 - e^{-(1-\epsilon)\log n}\right)^n\\
&= \left(1-\frac{1}{n^{1-\epsilon}}\right)^n = \left(\left(1-\frac{1}{n^{1-\epsilon}}\right)^{n^{1-\epsilon}}\right)^{n^\epsilon}
\end{aligned}$$

Using part (a) in the Borel-Cantelli lemma,

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\frac{Y_n}{\log n} < 1-\epsilon\right) &= \sum_{n=1}^{\infty}\left[\left(1-\frac{1}{n^{1-\epsilon}}\right)^{n^{1-\epsilon}}\right]^{n^\epsilon}\\
&\le \sum_{n=1}^{\infty}\left(e^{-1}\right)^{n^\epsilon} < \infty.
\end{aligned}$$

Therefore, $P\left(\frac{Y_n}{\log n} < 1-\epsilon \text{ i.o.}\right) = 0$. Using similar steps and focusing on the random variable X_n, we have

$$P\left(\frac{X_n}{\log n} > 1+\epsilon\right) = e^{-\log n(1+\epsilon)} = \frac{1}{n^{1+\epsilon}}.$$

Therefore,

$$\sum_{n=1}^{\infty} P\left(\frac{X_n}{\log n} > 1+\epsilon\right) = \sum_{n=1}^{\infty}\frac{1}{n^{1+\epsilon}} < \infty.$$

By part (a) in the Borel-Cantelli lemma, $P\left(\frac{X_n}{\log n} > 1+\epsilon \text{ i.o.}\right) = 0$. Also, $\sum_{n=1}^{\infty} P\left(\frac{X_n}{\log n} > 1-\epsilon\right) = \sum_{n=1}^{\infty}\frac{1}{n^{1-\epsilon}} = \infty$. By part (b) in the Borel-Cantelli lemma, $P\left(\frac{X_n}{\log n} > 1-\epsilon \text{ i.o.}\right) = 1$. Combining the above results,

$$\limsup_n \frac{X_n}{\log n} = 1 \text{ a.s.}$$

and therefore, $P\left(\frac{X_n}{\log n} > 1+\epsilon \text{ i.o.}\right) = 0$. Hence,

$$\frac{Y_n}{\log n} = \frac{\max_{1\le i\le n} X_i}{\log n} \xrightarrow[n\to\infty]{a.s.} 1.$$

Problem 4

1. By applying the Chernoff bound, we obtain:

$$P(X > t) = P\left(e^{uX} > e^{ut}\right) \leq \frac{\overbrace{E\left(e^{uX}\right)}^{m_X(u)}}{e^{ut}}, \tag{3.1}$$

which holds for any $u > 0$. By subgaussianity,

$$m_X(u) \leq e^{\sigma^2 u^2/2}. \tag{3.2}$$

Plugging (3.2) into (3.1) gives

$$P(X > t) \leq e^{\frac{\sigma^2 u^2}{2} - ut} = e^{\phi(u)}, \tag{3.3}$$

where $\phi(u) := \frac{\sigma^2 u^2}{2} - ut$. Choose $u_* = \frac{t}{\sigma^2}$, which minimizes $\phi(u)$ to obtain $\phi(u_*) = -\frac{t^2}{2\sigma^2}$. Therefore,

$$P(X > t) \leq e^{\phi(u_*)} = e^{-\frac{t^2}{2\sigma^2}}$$

as required.

2. Using the hint,

$$\begin{aligned}
E(\max_i X_i) &= \frac{1}{\lambda} E\left(\log\left(e^{\lambda \max_i X_i}\right)\right) \\
&\leq \frac{1}{\lambda} \log\left(E\left(e^{\lambda \max_i X_i}\right)\right) && \text{(Jensen, concavity of } \log(\cdot)) \\
&= \frac{1}{\lambda} \log\left(E\left(\max_i e^{\lambda X_i}\right)\right) && \text{(Monotonicity of } e^x) \\
&\leq \frac{1}{\lambda} \log\left(E\left(\sum_{i=1}^n e^{\lambda X_i}\right)\right) && \left(\max_i X_i \leq \sum_i X_i\right) \\
&\leq \frac{1}{\lambda} \log\left(n e^{\frac{\lambda^2 \sigma^2}{2}}\right) && \text{(subgaussianity)} \\
&= \underbrace{\frac{\log n}{\lambda} + \frac{\lambda \sigma^2}{2}}_{g(\lambda)}.
\end{aligned}$$

Minimizing $g(\lambda)$ by setting its derivative to zero, we obtain $\lambda_* = \frac{\sqrt{2\log n}}{2\sigma}$ for which we have

$$E[\max_i X_i] \leq \sigma\sqrt{2\log n}.$$

3. Invoking the union bound and part (a) we obtain:

$$P\left(\max_{1\leq i\leq n} X_i > t\right) = P\left(\bigcup_{i=1}^n \{X_i > t\}\right) \leq \sum_{i=1}^n \underbrace{P(X_i > t)}_{\leq e^{-\frac{t^2}{2\sigma^2}}} \leq n e^{-\frac{t^2}{2\sigma^2}}.$$

Problem 5

Let $X_n \xrightarrow[n\to\infty]{p} X$. Since $|X_n| \leq Y, \forall n$ we conclude that $|X| \leq Y$ almost surely. Moreover $V_n = |X_n - X| \leq |X_n| + |X| \leq 2Y$ almost surely. Choose $\epsilon > 0$ and fix it. By part (c) in Problem 7 we have $E[V_n] = E[V_n \mathbb{1}_{V_n \leq \epsilon}] + E[V_n \mathbb{1}_{V_n > \epsilon}] \leq E[\epsilon \mathbb{1}_{V_n \leq \epsilon}] + 2E[Y \mathbb{1}_{V_n > \epsilon}] \leq \epsilon + 2E[Y \mathbb{1}_{V_n > \epsilon}]$. By $X_n \xrightarrow[n\to\infty]{p} X$, we have that $P(V_n > \epsilon) \xrightarrow[n\to\infty]{} 0$. Using the assumption that $E[Y] < \infty$ and part (a), we conclude that $E[Y \mathbb{1}_{V_n > \epsilon}] \xrightarrow[n\to\infty]{} 0$. Letting now $\epsilon \to 0$ results in $E[V_n] \xrightarrow[n\to\infty]{} 0$.

Problem 6

Set A can be expressed as $A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{r=1}^{\infty} \{|X_{m+r} - X_m| \leq \epsilon_n\}$ for $\epsilon_n \xrightarrow[n\to\infty]{} 0$. Therefore, $A \in \mathcal{F}$. Furthermore, we can define the random variable $X : \Omega \to \mathbb{R}$ as $X(\omega) = \lim_n X_n(\omega)$ for $\omega \in A$ and 0 otherwise. Clearly, X is $\mathcal{F}$-measurable since $A \in \mathcal{F}$.

Problem 7

By the given hints, g is bounded and uniformly continuous on $[0, 1]$. Therefore, $\exists K > 0$ such that $|g(x)| \leq K, \forall x \in [0, 1]$ and also $\forall \epsilon > 0, \exists \delta > 0$ such that $|g(x) - g(\tilde{x})| < \epsilon$ for $|x - \tilde{x}| < \delta$ with $x, \tilde{x} \in [0, 1]$. By parts (b) and (c),

$$|E[X_n]| \leq E[|X_n| \mathbb{1}_C] + E[|X_n| \mathbb{1}_{C^c}] \leq \epsilon + 2KP(C^c) \leq \epsilon + 2K \frac{x(1-x)}{n\delta^2},$$

for any $\epsilon > 0$ and an associated $\delta > 0$. Here, Chebyshev's inequality has been used. For sufficiently large n, $2K \frac{x(1-x)}{n\delta^2} \leq \epsilon$, i.e, $|E[X_n]| = O(\epsilon)$. Since ϵ is arbitrary, the result follows.