Problem 1

The sequence has only two outcomes (or sample paths) depending on $X_1$:

$X_1 = 1:$ Then $X_1X_2\ldots = 10101010\ldots$

$X_1 = 0:$ Then $X_1X_2\ldots = 01010101\ldots$

Due to the oscillatory nature of the sample paths, it is easy to see that $\{X_n\}$ does not converge almost surely and in probability. Moreover, by Proposition 2.7 in the book, $\{X_n\}$ does not converge in m.s. sense. Finally, as $n \to \infty$, the value of $X_n$ is determined by $X_1$ and therefore, $X_n \sim \text{Ber}(\frac{1}{2})$, $\forall n \geq 1$ (due to $X_1$), hence $X_n \overset{d}{\to} \text{Ber}(\frac{1}{2})$.

Problem 2

Without loss of generality, take $X = 0$. We want to show that $X_n \overset{p}{\to} 0$ if and only if $\lim_{n \to \infty} E\left[\frac{|X_n|}{1+|X_n|}\right] = 0$.

(i) $X_n \overset{p}{\to} 0 \implies \lim_{n \to \infty} E\left[\frac{|X_n|}{1+|X_n|}\right] = 0.$

By $X_n \overset{p}{\to} 0$, we have that $\forall \epsilon > 0 : \lim_{n \to \infty} P(|X_n| > \epsilon) = 0$. Note that

$$\frac{|X_n|}{1+|X_n|} \leq \frac{|X_n|}{1+|X_n|} \mathbb{1}[|X_n| \leq \epsilon] + \epsilon \mathbb{1}[|X_n| \leq \epsilon] \leq \mathbb{1}[|X_n| > \epsilon] + \epsilon.$$

Therefore,

$$E\left[\frac{|X_n|}{1+|X_n|}\right] \leq E[\mathbb{1}[|X_n| > \epsilon]] + \epsilon = P(|X_n| > \epsilon) + \epsilon.$$

Taking the limit, we obtain $\lim_{n \to \infty} E\left[\frac{|X_n|}{1+|X_n|}\right] \leq \epsilon$, and since $\epsilon > 0$ is arbitrary, we have that $\lim_{n \to \infty} E\left[\frac{|X_n|}{1+|X_n|}\right] = 0$.

(ii) $\lim_{n \to \infty} E\left[\frac{|X_n|}{1+|X_n|}\right] = 0 \implies X_n \overset{p}{\to} 0.$

Observe that the function $f(x) = \frac{x}{1+x}$ is increasing. Therefore,

$$\frac{\epsilon}{1+\epsilon} \mathbb{1}[|X_n| > \epsilon] \leq \frac{|X_n|}{1+|X_n|} \mathbb{1}[|X_n| > \epsilon] \leq \frac{|X_n|}{1+|X_n|}.$$

3-1
Taking expectations and then limits to both sides, we obtain:

\[
\frac{\epsilon}{1+\epsilon} \lim_{n \to \infty} P(|X_n| > \epsilon) \leq \lim_{n \to \infty} E \left[ \frac{|X_n|}{|X_n| + 1} \right] = 0
\]

Since this holds for any \( \epsilon > 0 \), we have that \( \lim_{n \to \infty} P(|X_n| > \epsilon) = 0 \), \( \forall \epsilon > 0 \). Therefore, \( X_n \xrightarrow{p} 0 \).

**Problem 3**

Let \( Y_n = \max\{X_1, \ldots, X_n\} \). For \( \epsilon > 0 \), we have

\[
P \left( \frac{\max(X_1, \ldots, X_n)}{\log n} < 1 - \epsilon \right) = P \left( \frac{Y_n}{\log n} < 1 - \epsilon \right) = P \left( Y_n < (1 - \epsilon) \log n \right) = P \left( \cap_{i=1}^{n} \{ X_i < (1 - \epsilon) \log n \} \right) = \left( 1 - e^{-(1-\epsilon) \log n} \right)^n = \left( 1 - \frac{1}{n^{1-\epsilon}} \right)^n = \left( \left( 1 - \frac{1}{n^{1-\epsilon}} \right)^{n^{1-\epsilon}} \right)^n.
\]

Using part (a) in the Borel-Cantelli lemma,

\[
\sum_{n=1}^{\infty} P \left( \frac{Y_n}{\log n} < 1 - \epsilon \right) = \sum_{n=1}^{\infty} \left[ \left( 1 - \frac{1}{n^{1-\epsilon}} \right)^{n^{1-\epsilon}} \right] < \sum_{n=1}^{\infty} (\epsilon^{-1})^{n^{1-\epsilon}} < \infty.
\]

Therefore, \( P \left( \frac{Y_n}{\log n} < 1 - \epsilon \text{ i.o.} \right) = 0 \). Using similar steps and focusing on the random variable \( X_n \), we have

\[
P \left( \frac{X_n}{\log n} > 1 + \epsilon \right) = e^{-\log n (1+\epsilon)} = \frac{1}{n^{1+\epsilon}}.
\]

Therefore,

\[
\sum_{n=1}^{\infty} P \left( \frac{X_n}{\log n} > 1 + \epsilon \right) = \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty.
\]

By part (a) in the Borel-Cantelli lemma, \( P \left( \frac{X_n}{\log n} > 1 + \epsilon \text{ i.o.} \right) = 0 \). Also, \( \sum_{n=1}^{\infty} P \left( \frac{X_n}{\log n} > 1 - \epsilon \right) = \sum_{n=1}^{\infty} \frac{1}{n^{1-\epsilon}} = \infty \). By part (b) in the Borel-Cantelli lemma, \( P \left( \frac{X_n}{\log n} > 1 - \epsilon \text{ i.o.} \right) = 1 \). Combining the above results,

\[
\limsup_{n\to\infty} \frac{X_n}{\log n} = 1 \text{ a.s.}
\]

and therefore, \( P \left( \frac{X_n}{\log n} > 1 + \epsilon \text{ i.o.} \right) = 0 \). Hence,

\[
\frac{Y_n}{\log n} = \max_{1 \leq i \leq n} \frac{X_i}{\log n} \xrightarrow{a.s.} 1, \quad n \to \infty.
\]
Problem 4

1. By applying the Chernoff bound, we obtain:

\[ P(X > t) = P\left( e^{AX} > e^{ut} \right) \leq \frac{m_X(u)}{e^{ut}}, \]  

which holds for any \( u > 0 \). By subgaussianity,

\[ m_X(u) \leq e^{\sigma^2 u^2/2}. \]  

(3.2)

Plugging (3.2) into (3.1) gives

\[ P(X > t) \leq e^{\phi(u)} = e^{-\frac{t^2}{2\sigma^2}} \]  

(3.3)

where \( \phi(u) := \frac{\sigma^2 u^2}{2} - ut \). Choose \( u^* = \frac{t}{\sigma} \), which minimizes \( \phi(u) \) to obtain \( \phi(u^*) = -\frac{t^2}{2\sigma^2} \). Therefore,

\[ P(X > t) \leq e^{\phi(u^*)} = e^{-\frac{t^2}{2\sigma^2}} \]

as required.

2. Using the hint,

\[ E(\max_i X_i) = \frac{1}{\lambda} E\left( \log \left( e^{\lambda \max_i X_i} \right) \right) \]

\[ \leq \frac{1}{\lambda} \log \left( E \left( e^{\lambda \max_i X_i} \right) \right) \]  

(Jensen, concavity of \( \log(\cdot) \))

\[ = \frac{1}{\lambda} \log \left( E \left( \max_i e^{\lambda X_i} \right) \right) \]  

(Monotonicity of \( e^x \))

\[ \leq \frac{1}{\lambda} \log \left( E \left( \sum_{i=1}^n e^{\lambda X_i} \right) \right) \]  

\[ \leq \frac{1}{\lambda} \log \left( ne^{\frac{\lambda^2}{2}} \right) \]  

(subgaussianity)

\[ = \frac{\log n}{\lambda} + \frac{\lambda \sigma^2}{2}. \]

Minimizing \( g(\lambda) \) by setting its derivative to zero, we obtain \( \lambda^* = \sqrt{\frac{2 \log n}{\sigma^2}} \) for which we have

\[ E[\max_i X_i] \leq \sigma \sqrt{2 \log n}. \]

3. Invoking the union bound and part (a) we obtain:

\[ P\left( \max_{1 \leq i \leq n} X_i > t \right) = P\left( \bigcup_{i=1}^n \{ X_i > t \} \right) \leq \sum_{i=1}^n P(X_i > t) \leq ne^{-\frac{t^2}{2\sigma^2}} \leq e^{-\frac{t^2}{2\sigma^2}}. \]
Problem 5
Let $X_n \xrightarrow{p} X$. Since $|X_n| \leq Y, \forall n$ we conclude that $|X| \leq Y$ almost surely. Moreover $V_n = |X_n - X| \leq |X_n| + |X| \leq 2Y$ almost surely. Choose $\epsilon > 0$ and fix it. By part (c) in Problem 7 we have $E[V_n] = E[V_n 1_{V_n \leq \epsilon}] + E[V_n 1_{V_n > \epsilon}] \leq E[\epsilon 1_{V_n \leq \epsilon}] + 2E[Y 1_{V_n > \epsilon}]$. By $X_n \xrightarrow{p} X$, we have that $P(V_n > \epsilon) \xrightarrow{n \to \infty} 0$. Using the assumption that $E[Y] < \infty$ and part (a), we conclude that $E[Y 1_{V_n > \epsilon}] \xrightarrow{n \to \infty} 0$. Letting now $\epsilon \to 0$ results in $E[V_n] \xrightarrow{n \to \infty} 0$.

Problem 6
Set $A$ can be expressed as $A = \cap_{n=1}^\infty \bigcup_{m=1}^\infty \cap_{r=1}^\infty \{|X_m + r - X_m| \leq \epsilon_n\}$ for $\epsilon_n \xrightarrow{n \to \infty} 0$. Therefore, $A \in \mathcal{F}$. Furthermore, we can define the random variable $X : \Omega \to \mathbb{R}$ as $X(\omega) = \lim_n X_n(\omega)$ for $\omega \in A$ and 0 otherwise. Clearly, $X$ is $\mathcal{F}$-measurable since $A \in \mathcal{F}$.

Problem 7
By the given hints, $g$ is bounded and uniformly continuous on $[0, 1]$. Therefore, $\exists K > 0$ such that $|g(x)| \leq K, \forall x \in [0, 1]$ and also $\forall \epsilon > 0, \exists \delta > 0$ such that $|g(x) - g(\tilde{x})| < \epsilon$ for $|x - \tilde{x}| < \delta$ with $x, \tilde{x} \in [0, 1]$. By parts (b) and (c),

$$|E[X_n]| \leq E[|X_n 1_C|] + E[|X_n 1_{C^c}|] \leq \epsilon + 2K P(C^c) \leq \epsilon + 2K \frac{x(1-x)}{n \delta^2},$$

for any $\epsilon > 0$ and an associated $\delta > 0$. Here, Chebyshev’s inequality has been used. For sufficiently large $n$, $2K \frac{x(1-x)}{n \delta^2} \leq \epsilon$, i.e., $|E[X_n]| = O(\epsilon)$. Since $\epsilon$ is arbitrary, the result follows.