# ECE534, Spring 2020: Problem Set #3

Due Mar 27, 2020

1. **All Types of Convergence of Random Sequences**

   Consider a sequence of random variables $X_1, X_2, \ldots$ with its generic term defined as follows:
   $$X_1 \sim \text{Ber}(1/2) \text{ and } X_n = (X_{n-1} + 1) \bmod 2.$$
   Does this sequence converge almost surely, in probability, in mean square and in distribution? Justify your answer for each type of convergence separately.

2. **Convergence in Probability**

   Prove that $X_n \xrightarrow[n\to\infty]{p} X$ if and only if
   $$\lim_{n\to\infty} E\left[\frac{|X_n - X|}{1 + |X_n - X|}\right] = 0.$$

3. **Almost Sure Convergence [Extra Credit]**

   Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with $P(X_n > x) = e^{-x}, x \geq 0$. Show that
   $$\frac{\max\{X_1, X_2, \ldots, X_n\}}{\log n} \xrightarrow[n\to\infty]{\text{a.s.}} 1.$$

   **Note**: Use the Borel-Cantelli Lemma.

4. **Maximum of a Finite Set of sub-Gaussian Random Variables**

   A random variable $X \in \mathbb{R}$ is called *sub-Gaussian* with *variance proxy* $\sigma^2$ if $E[X] = 0$ and its moment generating function satisfies:
   $$m_X(u) = E\left[e^{uX}\right] \leq e^{\frac{u^2\sigma^2}{2}}, \quad \forall u \in \mathbb{R}.$$
   We write $X \sim \text{subG}(\sigma^2)$.

   (a) Use the Chernoff bound to show that
   $$P(X > t) \leq e^{-\frac{t^2}{2\sigma^2}}, \quad \forall t > 0$$
   when $X \sim \text{subG}(\sigma^2)$.

   (b) Let $X_1, X_2, \ldots, X_n$ be $\text{subG}(\sigma^2)$ random variables, not necessarily independent. Show that the expectation of the maximum can be bounded as
   $$E[\max_{1\leq i\leq n} X_i] \leq \sigma\sqrt{2\log n}.$$

   **Hint**: Start your derivation by noting that
   $$E[\max_{1\leq i\leq n} X_i] = \frac{1}{\lambda} E\left[\log\left(e^{\lambda \max_{1\leq i\leq n} X_i}\right)\right], \quad \forall \lambda > 0.$$

(c) With the same assumptions as in the previous part, show that

$$P\left(\max_{1\leq i\leq n} X_i > t\right) \leq ne^{-\frac{t^2}{2\sigma^2}}, \quad \forall t > 0.$$

5. **Dominated Convergence**

Suppose that $|X_n| \leq Y, \forall n \geq 1$ and $E[Y] < \infty$. Show that if $X_n \xrightarrow[n\to\infty]{p} X$, then $E[|X_n - X|] \xrightarrow[n\to\infty]{} 0$ (convergence of $X_n$ *in mean* to $X$ or in the $L^1$ norm).

**Notes**:

(a) For a random variable $Z$ the following holds: $E[|Z|] < \infty$ if and only if $\forall \epsilon > 0, \exists \delta > 0$ such that $E[|Z|\mathbb{1}_A] < \epsilon$ for all events $A$ such that $P(A) < \delta$.

(b) Define $V_n = |X_n - X|$. Start you derivation using Note $(b)$ in Problem 7 for $C = \{V_n \leq \epsilon\}$, where $\epsilon > 0$ is fixed. Conclude your derivation using part (a) by allowing $\epsilon$ to decrease to zero.

6. **Sequences of Random Variables and Convergence Sets**

Let $(X_n)_{n\geq 1}$ be a sequence of random variables on $(\Omega, \mathcal{F}, P)$. Consider the set $A = \{\omega \in \Omega : (X_n(\omega))_{n\geq 1}$ converges$\}$. Show that $A \in \mathcal{F}$ (i.e., $A$ is an event) and that there exists a random variable $X : \Omega \to \mathbb{R}$, i.e., an $\mathcal{F}$-measurable mapping, such that $X_n(\omega) \to X(\omega), \forall \omega \in A$.

**Hint**: To show that $A \in \mathcal{F}$, try to express it in terms of countable unions and intersections. Based on this result, identify $X$.

7. **Polynomial Approximation of Continuous Functions [Extra Credit]**

Consider a continuous mapping $g : [0,1] \to \mathbb{R}$. Show that

$$\lim_{n\to\infty} \sup_{0\leq x\leq 1} \left|\sum_{k=0}^{n} g\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k} - g(x)\right| = 0.$$

**Notes**:

(a) The above statement corresponds to the so-called *Weierstrass Approximation Theorem*. It states that every continuous function $g$ on a closed interval can be approximated *uniformly* over this interval by a polynomial.

(b) Start you derivation by letting $Y_n \sim \text{Bin}(n, x)$. Define the random variable $X_n = g(x) - g\left(\frac{Y_n}{n}\right)$.

(c) Note that for every random variable $V$, $E[V] = E[V\mathbb{1}_C] + E[V\mathbb{1}_{C^c}]$ for an event $C$. In the context of this problem, define $V = X_n$ and $C = \{\left|\frac{Y_n}{n} - x\right| > \delta\}$. The goal is then to show that $|E[X_n]| = O(\epsilon)$ for a sufficiently large $n$. $P(C)$ can be bounded using known inequalities.

(d) Every continuous function $g$ on a closed interval is *bounded*.

(e) $g$ uniformly continuous on $[0,1]$: $\forall \epsilon > 0, \exists \delta > 0$ such that $|g(x) - g(\tilde{x})| < \epsilon$ for $|x - \tilde{x}| < \delta$ with $x, \tilde{x} \in [0,1]$.