ECE534 Random Processes Spring 2020	University of Illinois at Urbana-Champaign Additional Probability Concepts

HW 2: Solution

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Problem 1

Since A, B are independent events

$$P(AB) = P(A)P(B) \quad \text{or} \quad \frac{|AB|}{p} = \frac{|A|}{p}\frac{|B|}{p},$$

i.e., $|AB|p = |A||B|$. Therefore, p is a divisor of $|A||B|$, assuming that $|A||B| > 0$. Since p is prime, p is a divisor of at least one of $|A|$ and $|B|$ (this elementary fact is the so-called *Euclid's Lemma* in number theory) and because $p \geq \max\{|A|, |B|\}$ the conclusion follows.

Problem 2

By the hint, it is easy to see that $P(\mathbb{1}_R(e) = 1) = \frac{1}{2}, \forall e \in E$. Therefore, $E[N_R] = \frac{|E|}{2}$. This implies that there exists at least one $R \subseteq V$ such that $N_R \geq \frac{|E|}{2}$.

Problem 3

By conditioning on the first toss we obtain

$$p_{kr} = pp_{k-1,r} + (1-p)p_{k,r-1}.$$

Additionally, $p_{k0} = 0$ and $p_{0r} = 1$ for $k, r \geq 1$.

Problem 4

By the provided hint $P(Z_i = 1) = P(R_i = 1)P(Y_i = 1) = \frac{p_1}{p_2}p_2 = p_1, \forall i$. Therefore, the Z_i's are i.i.d. and have the same distribution with the X_i's. This implies that $Z = (Z_1, Z_2, \ldots, Z_n)$ and $X = (X_1, X_2, \ldots, X_n)$ are identically distributed vectors. Moreover, by construction, $Z_i \leq Y_i, \forall i$. Therefore, $Z \leq Y$ and since f is increasing, $f(Z) \leq f(Y)$, for every realization of Z, Y. By taking expectations to both sides, we conclude that $E[f(Z)] = E[f(X)] \leq E[f(Y)]$.

Problem 5

For the first part, note that any real number x can be written as

$$x = \int_0^x 1dt = \int_0^\infty \mathbb{1}(x > t)dt.$$

By replacing x with X and by taking the expectation to both sides the first identity follows. To exchange the order of expectation and integration we employ the *Fubini-Tonelli theorem*.

For the second part, we use the first part to write

$$E[|Y|^p] = \int_0^\infty P(|Y|^p > \tilde{t})d\tilde{t} = \int_0^\infty P\left(|Y| > \sqrt[p]{\tilde{t}}\right)d\tilde{t}.$$

By changing variables to $t = \sqrt[p]{\tilde{t}}$ and by noting that $pt^{p-1}dt = d\tilde{t}$ the second identity follows.

Problem 6

By conditioning on any of the variables, say on X_n, the recursion in the hint follows. Clearly, $Q_0(x) = 1, \forall x \in (0, 1]$. This shows that $Q_1(x) = x, Q_2(x) = \frac{x^2}{2!}, \ldots, Q_n(x) = \frac{x^n}{n!}$ (can be verified inductively). By employing the note in the previous problem, $E[N] = \sum_{n=0}^{\infty} P(N > n) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

Problem 7

(a)

$$E\left[e^{\lambda x}\right] = \frac{1}{2}\left[e^{\lambda} + e^{-\lambda}\right] = \frac{1}{2}\left[\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} + \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\right] = \sum_{k=0}^{\infty}\frac{\lambda^{2k}}{(2k)!}$$
$$\leq 1 + \sum_{k=1}^{\infty}\frac{\lambda^{2k}}{2^k k!} = 1 + \sum_{k=1}^{\infty}\frac{(\lambda^2/2)^k}{k!} = e^{\frac{\lambda^2}{2}},$$

where we have used the observation that $(2k)! = k!(k+1)\cdots(2k) \geq k!2^k, \forall k \geq 1$.

(b) i.

$$E_X\left[e^{\lambda X}\right] = E_X\left[e^{\lambda(X - E_{X'}[X'])}\right] \leq E_{X,X'}\left[e^{\lambda(X-X')}\right],$$

where $E[X'] = 0$ and Jensen's inequality have been used.

ii.

$$E_{X,X'}\left[e^{\lambda(X-X')}\right] = E_{X,X',\varepsilon}\left[e^{\lambda\varepsilon(X-X')}\right] = E_{X,X'}\left[E_{\varepsilon}\left[e^{\lambda\varepsilon(X-X')}\right]\right]$$
$$\leq E_{X,X'}\left[e^{\frac{\lambda^2(X-X')^2}{2}}\right].$$

Since $X \in [a, b]$, we have that $|X - X'| \leq (b-a)$ and therefore, $E_{X,X'}\left[e^{\frac{\lambda^2(X-X')^2}{2}}\right] \leq E_{X,X'}\left[e^{\frac{\lambda^2(b-a)^2}{2}}\right] = e^{\frac{\lambda^2(b-a)^2}{2}}$. Hence, $E\left[e^{\lambda X}\right] \leq e^{\frac{\lambda^2(b-a)^2}{2}}$.
This bound has been sharpened in class via Hoeffding's Lemma.

Problem 8

$$E[M] = \frac{1}{n}\sum_{i=1}^{n} E\left[\frac{g(Y_i)f_X(Y_i)}{f_Y(Y_i)}\right] = \frac{1}{n} n E\left[\frac{g(Y)f_X(Y)}{f_Y(Y)}\right]$$
$$= \int \frac{g(y)f_X(y)}{f_Y(y)} f_Y(y)dy = \int g(x)f_X(x)dx,$$

where the last part is due to the equivalence of f_X, f_Y.

For the variance,

$$\text{Var}(M) = \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{g(Y_i)f_X(Y_i)}{f_Y(Y_i)}\right) = \frac{1}{n^2} n \text{Var}\left(\frac{g(Y)f_X(Y)}{f_Y(Y)}\right)$$
$$= \frac{1}{n}\left(E\left[\frac{g^2(Y)f_X^2(Y)}{f_Y^2(Y)}\right] - (E[g(X)])^2\right).$$

Here, we have used the fact that $\frac{g(Y_i)f_X(Y_i)}{f_Y(Y_i)}, i = 1, 2, \ldots, n$ are i.i.d. random variables and the result for $E[M]$.