ECE534, Spring 2020: Problem Set #2

Due Feb 26, 2020

1. **Independence of Events and Number Theory**
   Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space such that Ω is finite with cardinality $|\Omega| = p$ where p is a prime number, $\mathcal{F} = 2^{\Omega}$ is the power set of Ω and $\mathbb{P}(A) = |A|/|\Omega| = |A|/p$ is the probability of any event $A \in \mathcal{F}$. Let A, B be independent events. Prove that at least one of these two events is either $\emptyset$ or Ω.

2. **Graphs and Random Sampling**
   Let $G = (V, E)$ be a graph with node set V and edge set E, both finite. For an arbitrary set $R \subseteq V$ and $e \in E$ define the indicator variable

   $$\mathbb{1}_R(e) = \begin{cases} 1, & \text{if } e \text{ connects } R \text{ and } R^c, \\ 0, & \text{otherwise} \end{cases}.$$

   Let $N_R = \sum_{e \in E} \mathbb{1}_R(e)$ be the number of edges connecting R, R^c ((R, R^c) is a *cut* of G and N_R corresponds to the cardinality of the associated *cut-set*). Show that $\exists R \subseteq V$ such that $N_R \geq |E|/2$.

   Hint: Assign to each $v \in V$ a Bernoulli random variable with probability of success $1/2$. Randomly sample V by placing in R all v's with Bernoulli values 1. Compute EN_R and establish from this computation the desired conclusion. This line of reasoning is an instance of the so-called **probabilistic method**.

3. **Probability Recursions**
   Consider a coin with $P(H) = p$. The coin is tossed repeatedly. Player A wins if k heads appear before r tails. Otherwise player B wins. Let $P(A \text{ wins}) = p_{kr}$. Find a recursion for p_{kr} and the associated boundary conditions, i.e., the values for p_{k0}, p_{0r}.

4. **A precursor inequality on finite distributive lattices**
   Let $f : \{0,1\}^n \to \mathbb{R}$ be an increasing function, i.e., $f(x) \leq f(y)$ for $x = (x_1, \ldots, x_n) \leq y = (y_1, \ldots, y_n)$, where $\leq$ for vectors denotes elementwise ordering (i.e., $x_i \leq y_i, i = 1, 2, \ldots, n$). Let $X = (X_1, X_2, \ldots, X_n), Y = (Y_1, Y_2, \ldots, Y_n)$ be vectors of i.i.d. $\text{Ber}(p_1)$ and $\text{Ber}(p_2)$ random variables, respectively. More explicitly, $X_i \sim \text{Ber}(p_1), i = 1, 2, \ldots, n$ and $Y_i \sim \text{Ber}(p_2), i = 1, 2, \ldots, n$. Suppose that $p_1 \leq p_2$. Show that $E[f(X)] \leq E[f(Y)]$.

   Hint: Let $R = (R_1, \ldots, R_n)$ be a vector of i.i.d. $\text{Ber}(p_1/p_2)$ random variables, independent of everything else. Define $Z_i = R_i Y_i, i = 1, \ldots, n$ and note that the Z_i's are i.i.d. Bernoulli random variables which are identically distributed with the X_i's.

5. **Classical Identities [Extra Credit]**
   Let X be a nonnegative random variable. Show that

   $$E[X] = \int_0^\infty P(X > t)dt.$$

   Extend this inequality to show that for any $p \in (0, \infty)$ and any random variable Y

   $$E[|Y|^p] = \int_0^\infty pt^{p-1} P(|Y| > t)dt,$$

whenever the right-hand side is finite.

Note: For a nonnegative integer-valued random variable T such that $E[T] < \infty$, the corresponding identity is $E[T] = \sum_{n=1}^{\infty} P(T \geq n) = \sum_{n=0}^{\infty} P(T > n)$.

6. **Integral Recursions**
   Let $(X_k)_{k \geq 1}$ be a sequence of i.i.d. random variables uniformly distributed over $[0, 1]$. For $x \in (0, 1)$ define the random variable

   $$N = \min \{n \geq 1 : X_1 + X_2 + \cdots + X_n > x\}.$$

   Let $Q_n(x) = P(N > n)$. Prove that $Q_n(x) = x^n/n!$ and use the note in Problem 5 to compute $E[N]$.

   Hint: Argue that $Q_n(x)$ satisfies the recursion $Q_n(x) = \int_0^x Q_{n-1}(x-u)du$. Initialize the recursion appropriately (i.e., choose the value of $Q_0(x)$ for any $x \in (0, 1]$) and solve the recursion to obtain $Q_n(x)$.

7. **Rademacher Random Variables and Symmetrization**

   (a) Let X be a Rademacher random variable, i.e., $P(X = \pm 1) = 1/2$. Show that

   $$E\left[e^{\lambda X}\right] \leq e^{\lambda^2/2}.$$

   (b) Let X be a zero mean random variable supported on the interval $[a, b]$.

   i. Assume that X' is an independent copy of X, i.e., X, X' are i.i.d. random variables. Using Jensen's inequality, show that

   $$E\left[e^{\lambda X}\right] \leq E_{X,X'}\left[e^{\lambda(X-X')}\right],$$

   where $E_{X,X'}[\cdot]$ denotes expectation with respect to the joint distribution of X, X'.

   ii. Assume that ε is a Rademacher random variable, independent of X, X'. Observe that the random variables $X - X'$ and $\varepsilon(X - X')$ have the same distribution, which is symmetric. Use the tower property of conditional expectation and the inequality in the previous part to show that

   $$E\left[e^{\lambda X}\right] \leq e^{\lambda^2(b-a)^2/2}.$$

8. **Basics of Importance Sampling**
   Let $Z = g(X)$. Suppose that we want to estimate the mean value of Z, which is given by $E[Z] = \int g(x) f_X(x)dx$. Assume that it is difficult to draw samples from f_X or that $g(X)$ has a very large variance. Let f_Y be an *equivalent* density to f_X for all x, i.e., $f_Y(x) = 0$ if and only if $f_X(x) = 0$. Let $\{Y_i\}_{i=1}^n$ be i.i.d. random variables with density f_Y. Define $M = \frac{1}{n}\sum_{i=1}^n \frac{g(Y_i) f_X(Y_i)}{f_Y(Y_i)}$. Prove that $E[M] = E[g(X)] = E\left[\frac{g(Y) f_X(Y)}{f_Y(Y)}\right]$ and $\text{Var}(M) = \frac{1}{n}\left(E\left[\frac{g^2(Y) f_X^2(Y)}{f_Y^2(Y)}\right] - (E[g(X)])^2\right)$.

   Note: f_Y is called *importance density*. The underlying key idea is that we can easily sample from f_Y. Moreover, f_Y is usually chosen so that $\text{Var}(M)$ is smaller than $\frac{1}{n}\left(E\left[g^2(X)\right] - (E[g(X)])^2\right)$, which is the variance of $\tilde{M} = \frac{1}{n}\sum_{i=1}^n g(X_i)$.