ECE534 Random Processes | **University of Illinois at Urbana-Champaign**
Spring 2020 | **Basic Probability Concepts**

HW 1: Solution

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Problem 1

For this problem,

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

and $A = \{HHH, HHT, HTH, HTT\}$, $B = \{HHH, HHT, THH, THT\}$, $C = \{HHT, THH\}$. Clearly,

$$P(A) = P(B) = \frac{1}{2}$$

and

$$P(C) = \frac{1}{4}.$$

Furthermore,

$$\begin{aligned} P(AB) &= P(\{HHH, HHT\}) = \frac{1}{4} = P(A)P(B), \\ P(AC) &= P(\{HHT\}) = \frac{1}{8} = P(A)P(C), \\ P(BC) &= P(\{HHT, THH\}) = \frac{1}{4} \neq P(B)P(C). \end{aligned}$$

Hence, A, B, C are not independent.

Problem 2

By the event axioms

(a) $A \cap B = (A^c \cup B^c)^c$, therefore $A \cap B \in \mathcal{F}$.

(b) $A \setminus B = A \cap B^c$ and by the previous part, $A \setminus B \in \mathcal{F}$.

(c) $A \triangle B = (A \setminus B) \cup (B \setminus A) \in \mathcal{F}$ by the previous part.

Problem 3

The proof is by induction. For $n = 2$, the principle holds due to $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 A_2)$. Suppose that the result holds for any $n \leq r$. For $n = r+1$, we have that $P\left(\bigcup_{i=1}^{r+1} A_i\right) = P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left(\left(\bigcup_{i=1}^{r} A_i\right) \cap A_{r+1}\right) = P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left(\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right)$ and the result follows easily by invoking the induction hypothesis.

Problem 4

We have

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{k\to\infty} P\left(\bigcap_{n=1}^{k} A_n\right) = \lim_{k\to\infty} P\left(\left(\bigcup_{n=1}^{k} A_n^c\right)^c\right)$$
$$= \lim_{k\to\infty}\left[1 - P\left(\bigcup_{n=1}^{k} A_n^c\right)\right] \geq \lim_{k\to\infty}\left[1 - \sum_{n=1}^{k} P\left(A_n^c\right)\right] = 1.$$

Problem 5

Let $Z = X + Y$. By relying on the independence of X, Y we have

$$\begin{aligned} P(Z = n) = P(X + Y = n) &= \sum_{j=0}^{n} p_{XY}(X = j, Y = n - j) \\ &= \sum_{j=0}^{n} p_X(X = j) p_Y(Y = n - j) \\ &= \sum_{j=0}^{n} e^{-\lambda_1}\frac{\lambda_1^j}{j!} e^{-\lambda_2}\frac{\lambda_2^{n-j}}{(n-j)!} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}\lambda_1^j\lambda_2^{n-j} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n. \end{aligned}$$

Therefore, $Z \sim \text{Pois}(\lambda_1 + \lambda_2)$.

Problem 6

1. By solving $1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x, y)dxdy = \int_0^1\int_0^{1-x} cxydydx$ for c, we obtain $c = 24$.
2. $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy = \int_0^{1-x} 24xydy = 12x(1-x)^2$ for $0 \leq x \leq 1$ and 0 otherwise.
3. $P(X + Y < 1/2) = \int_0^{1/2}\int_0^{1/2-x} 24xydydx = \frac{1}{16}$.
4. The support of f_{XY} is not a product set, hence X, Y are not independent.

Problem 7

For $k \leq n$,

$$E\left[\frac{\sum_{i=1}^k X_i}{\sum_{j=1}^n X_j}\right] = \sum_{i=1}^{k} E\left[\frac{X_i}{\sum_{j=1}^n X_j}\right] = kE\left[\frac{X_1}{\sum_{j=1}^n X_j}\right],$$

where in the last step we employ the observation that

$$\frac{X_1}{\sum_{j=1}^n X_j}, \frac{X_2}{\sum_{j=1}^n X_j}, \ldots, \frac{X_k}{\sum_{j=1}^n X_j}$$

are identically distributed random variables due to the i.i.d. assumption on $X_1, X_2, \dots, X_n$ and therefore, they have the same mean value. Moreover, for $k = n$,

$$1 = E\left[\frac{\sum_{i=1}^n X_i}{\sum_{j=1}^n X_j}\right] = nE\left[\frac{X_1}{\sum_{j=1}^n X_j}\right].$$

Hence,

$$E\left[\frac{\sum_{i=1}^k X_i}{\sum_{j=1}^n X_j}\right] = \frac{k}{n}, \ \ 1 \le k \le n.$$

Problem 8

By relying on the total law of probability we can write

$$\begin{aligned}
P(Y > X) &= \sum_{j=0}^{\infty} P(Y > X | X = j) p_j \\
&= \sum_{j=0}^{\infty} P(Y > j | X = j) p_j \\
&= \sum_{j=0}^{\infty} P(Y > j) p_j \quad \text{(independence of } X, Y\text{)} \\
&= \sum_{j=0}^{\infty} (1-p)^j p_j \quad \text{(tail property of } \mathrm{Geo}(p)\text{)} \\
&= \sum_{j=0}^{\infty} z^j p_j = G(z).
\end{aligned}$$

Problem 9

It is easy to see that

$$a^r \mathbb{1}(|Y| \ge a) + (b^r - a^r)\mathbb{1}(|Y| \ge b) \le |Y|^r.$$

By taking the expectations to both sides the result follows.

Problem 10

1. The order statistics $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ will take the values $x_1 \le x_2 \le \cdots \le x_n$ if and only if there exists a permutation $\{\pi(1), \pi(2), \dots \pi(n)\}$ of $\{1, 2, \dots, n\}$ such that $X_1 = x_{\pi(1)}, X_2 = x_{\pi(2)}, \dots, X_n = x_{\pi(n)}$. For any permutation π we have

$$\begin{aligned}
&P\left(x_{\pi(1)} - \frac{\Delta x}{2} < X_1 < x_{\pi(1)} + \frac{\Delta x}{2}, \dots, x_{\pi(n)} - \frac{\Delta x}{2} < X_n < x_{\pi(n)} + \frac{\Delta x}{2}\right) \\
&\approx (\Delta x)^n f_{X_1,\dots,X_n}\left(x_{\pi(1)}, \dots, x_{\pi(n)}\right) = (\Delta x)^n \prod_{i=1}^n f(x_i).
\end{aligned}$$

   Therefore, for any $x_1 < x_2 < \cdots < x_n$,

$$\begin{aligned}
&P\left(x_1 - \frac{\Delta x}{2} < X_{(1)} < x_1 + \frac{\Delta x}{2}, \dots, x_n - \frac{\Delta x}{2} < X_{(n)} < x_n + \frac{\Delta x}{2}\right) \\
&\approx n!(\Delta x)^n \prod_{i=1}^n f(x_i).
\end{aligned}$$

Dividing by $(\Delta x)^n$ and letting $\Delta x \to 0$ yields the result.

2. The probability to be computed can be equivalently expressed in terms of the order statistics $X_{(1)} \leq X_{(2)} \leq X_{(3)}$ of the point positions as $P\left(X_{(3)} \geq X_{(2)} + d, X_{(2)} \geq X_{(1)} + d\right)$. By employing part (a)

$$\begin{aligned} P\left(X_{(3)} \geq X_{(2)} + d, X_{(2)} \geq X_{(1)} + d\right) &= \int_0^{1-2d} \int_{x_1+d}^{1-d} \int_{x_2+d}^{1} 3! dx_3 dx_2 dx_1 \\ &= (1-2d)^3. \end{aligned}$$