Problem 1
For this problem,
\[ \Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}, \]
and \( A = \{HHH, HHT, HTH, HTT\}, B = \{HHH, HHT, THH, THT\}, C = \{HHT, THH\} \). Clearly,
\[ P(A) = P(B) = \frac{1}{2} \]
and
\[ P(C) = \frac{1}{4}. \]
Furthermore,
\[ P(AB) = P(\{HHH, HHT\}) = \frac{1}{4} = P(A)P(B), \]
\[ P(AC) = P(\{HHT\}) = \frac{1}{8} = P(A)P(C), \]
\[ P(BC) = P(\{HHT, THH\}) = \frac{1}{4} \neq P(B)P(C). \]
Hence, \( A, B, C \) are not independent.

Problem 2
By the event axioms
\begin{enumerate}
  \item \( A \cap B = (A^c \cup B^c)^c \), therefore \( A \cap B \in \mathcal{F} \).
  \item \( A \setminus B = A \cap B^c \) and by the previous part, \( A \setminus B \in \mathcal{F} \).
  \item \( A \triangle B = (A \setminus B) \cup (B \setminus A) \in \mathcal{F} \) by the previous part.
\end{enumerate}

Problem 3
The proof is by induction. For \( n = 2 \), the principle holds due to
\[ P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1A_2). \]
Suppose that the result holds for any \( n \leq r \). For \( n = r + 1 \), we have that
\[ P(\bigcup_{i=1}^{r+1} A_i) = P(\bigcup_{i=1}^{r} A_i) + P(A_{r+1}) - P(\bigcup_{i=1}^{r+1} A_i \cap A_{r+1}) = P(\bigcup_{i=1}^{r} A_i) + P(A_{r+1}) - P(\bigcup_{i=1}^{r} (A_i \cap A_{r+1})) \]
and the result follows easily by invoking the induction hypothesis.
Problem 4
We have
\[
P \left( \bigcap_{n=1}^{\infty} A_n \right) = \lim_{k \to \infty} P \left( \bigcap_{n=1}^{k} A_n \right) = \lim_{k \to \infty} P \left( \bigcup_{n=1}^{k} A_n^c \right)^c
\]
\[
= \lim_{k \to \infty} \left[ 1 - P \left( \bigcup_{n=1}^{k} A_n^c \right) \right] \geq \lim_{k \to \infty} \left[ 1 - \sum_{n=1}^{k} P (A_n^c) \right] = 1.
\]

Problem 5
Let \( Z = X + Y \). By relying on the independence of \( X, Y \) we have
\[
P(Z = n) = P(X + Y = n) = \sum_{j=0}^{n} p_{XY}(X = j, Y = n - j)
\]
\[
= \sum_{j=0}^{n} p_X(X = j)p_Y(Y = n - j)
\]
\[
= \sum_{j=0}^{n} e^{-\lambda_1} \frac{\lambda_1^j}{j!} e^{-\lambda_2} \frac{\lambda_2^{n-j}}{(n-j)!}
\]
\[
= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \lambda_1^j \lambda_2^{n-j}
\]
\[
= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1 + \lambda_2)^n.
\]

Therefore, \( Z \sim \text{Pois}(\lambda_1 + \lambda_2) \).

Problem 6
1. By solving \( 1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{XY}(x, y) dx dy = \int_{0}^{1} \int_{0}^{1-x} cxy dy dx \) for \( c \), we obtain \( c = 24 \).
2. \( f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) dy = \int_{0}^{1-x} 24 xy dy = 12x(1 - x)^2 \) for \( 0 \leq x \leq 1 \) and 0 otherwise.
3. \( P(X + Y < 1/2) = \int_{0}^{1/2} \int_{0}^{1/2-x} 24 xy dy dx = \frac{1}{18} \).
4. The support of \( f_{XY} \) is not a product set, hence \( X, Y \) are not independent.

Problem 7
For \( k \leq n \),
\[
E \left[ \frac{\sum_{i=1}^{k} X_i}{\sum_{j=1}^{n} X_j} \right] = \sum_{i=1}^{k} E \left[ \frac{X_i}{\sum_{j=1}^{n} X_j} \right] = k E \left[ \frac{X_1}{\sum_{j=1}^{n} X_j} \right],
\]
where in the last step we employ the observation that
\[
\frac{X_1}{\sum_{j=1}^{n} X_j}, \frac{X_2}{\sum_{j=1}^{n} X_j}, \ldots, \frac{X_k}{\sum_{j=1}^{n} X_j}
\]
are identically distributed random variables due to the i.i.d. assumption on \(X_1, X_2, \ldots, X_n\) and therefore, they have the same mean value. Moreover, for \(k = n\),

\[
1 = E \left[ \frac{\sum_{i=1}^{n} X_i}{\sum_{j=1}^{n} X_j} \right] = nE \left[ \frac{X_1}{\sum_{j=1}^{n} X_j} \right].
\]

Hence,

\[
E \left[ \frac{\sum_{i=1}^{k} X_i}{\sum_{j=1}^{n} X_j} \right] = \frac{k}{n}, \quad 1 \leq k \leq n.
\]

**Problem 8**

By relying on the total law of probability we can write

\[
P(Y > X) = \sum_{j=0}^{\infty} P(Y > X|X = j)p_j = \sum_{j=0}^{\infty} P(Y > j|X = j)p_j = \sum_{j=0}^{\infty} P(Y > j)p_j \quad \text{(independence of } X, Y) = \sum_{j=0}^{\infty} (1 - p)^j p_j \quad \text{(tail property of Geo}(p)) = \sum_{j=0}^{\infty} z^j p_j = G(z).
\]

**Problem 9**

It is easy to see that

\[
a^r \mathbb{1}(|Y| \geq a) + (b^r - a^r) \mathbb{1}(|Y| \geq b) \leq |Y|^{r}.
\]

By taking the expectations to both sides the result follows.

**Problem 10**

1. The order statistics \(X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}\) will take the values \(x_1 \leq x_2 \leq \cdots \leq x_n\) if and only if there exists a permutation \(\{\pi(1), \pi(2), \ldots, \pi(n)\}\) of \(\{1, 2, \ldots, n\}\) such that \(X_1 = x_{\pi(1)}, X_2 = x_{\pi(2)}, \ldots, X_n = x_{\pi(n)}\). For any permutation \(\pi\) we have

\[
P \left( x_{\pi(1)} - \frac{\Delta x}{2} < X_1 < x_{\pi(1)} + \frac{\Delta x}{2}, \ldots, x_{\pi(n)} - \frac{\Delta x}{2} < X_n < x_{\pi(n)} + \frac{\Delta x}{2} \right) \approx (\Delta x)^n f_{X_1, \ldots, X_n}(x_{\pi(1)}, \ldots, x_{\pi(n)}) = (\Delta x)^n \prod_{i=1}^{n} f(x_i).
\]

Therefore, for any \(x_1 < x_2 < \cdots < x_n\),

\[
P \left( x_1 - \frac{\Delta x}{2} < X_1 < x_1 + \frac{\Delta x}{2}, \ldots, x_n - \frac{\Delta x}{2} < X_n < x_n + \frac{\Delta x}{2} \right) \approx n!(\Delta x)^n \prod_{i=1}^{n} f(x_i).
\]
Dividing by \((\Delta x)^n\) and letting \(\Delta x \to 0\) yields the result.

2. The probability to be computed can be equivalently expressed in terms of the order statistics \(X_{(1)} \leq X_{(2)} \leq X_{(3)}\) of the point positions as \(P\left(X_{(3)} \geq X_{(2)} + d, X_{(2)} \geq X_{(1)} + d\right)\). By employing part (a)

\[
P\left(X_{(3)} \geq X_{(2)} + d, X_{(2)} \geq X_{(1)} + d\right) = \int_0^{1-2d} \int_{x_1+d}^{1-d} \int_{x_2+d}^{1} 3!dx_3 dx_2 dx_1
= (1 - 2d)^3.
\]