ECE534, Spring 2020: Problem Set #1

Due Feb 17, 2020

1. **Independence of Events**
   Suppose that a fair coin (yielding either H for "heads" or T for "tails") is tossed three times. Define the events:

   - A: H appears on the first toss.
   - B: H appears on the second toss.
   - C: Exactly two H's are tossed in a row.

   Are the events A, B and C independent?

2. **Basics of σ-algebras**
   Let A, B be two events in some σ-field $\mathcal{F}$. Argue that $A \cap B$, $A \setminus B$ and $A \triangle B = (A \setminus B) \cup (B \setminus A)$ belong to $\mathcal{F}$. Here, $A \setminus B$ denotes set difference.

3. **Inclusion-Exclusion Principle**
   Let $A_1, A_2, \ldots, A_n, n \geq 2$ be events. Prove that

   $$P\left(\bigcup_{i=1}^{n} A_n\right) = \sum_i P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) - \cdots + (-1)^{n+1} P(A_1 A_2 \cdots A_n)$$

4. **Continuity of Probability Measure and Sequences of Events**
   Let $\{A_n\}_{n \geq 1}$ be a sequence of events such that $P(A_n) = 1, \forall n$. Use the continuity of probability measure and the union bound to show that $P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1$.

5. **Sum of Independent Random Variables**
   Let $X \sim \text{Pois}(\lambda_1)$ and $Y \sim \text{Pois}(\lambda_2)$ be independent Poisson random variables. Compute the distribution of $X + Y$.

6. **Bivariate Densities Revision**
   Let X, Y be two random variables with joint density function

   $$f_{XY}(u, v) = \begin{cases} cuv, & 0 \leq u \leq 1, 0 \leq v \leq 1, u + v \leq 1 \\ 0, & \text{else} \end{cases}$$

   for some constant $c > 0$.

   (a) Find c such that f_{XY} is a valid probability density function.
   (b) Find the marginal f_X.
   (c) Compute $P(X + Y < 1/2)$.
   (d) Are X, Y independent? Justify your answer.

7. **Expected Ratios**
   Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed positive random variables. Compute

   $$E\left[\frac{\sum_{i=1}^{k} X_i}{\sum_{i=1}^{n} X_i}\right], \quad k \leq n.$$

8. **Probability Generating Functions**
The probability generating function of a nonnegative integer-valued random variable X is defined as $G(z) = E[z^X] = \sum_{j=0}^{\infty} p_j z^j$, where $\{p_j\}_{j\geq 0}$ is the probability mass function of X and $z \in \mathbb{C}$. Let $0 < z < 1$ and assume that $Y \sim \text{Geo}(p)$, where $p = 1 - z$. Suppose that Y is independent of X. Show that $G(z) = P(X < Y)$.

9. **Some Tail Bound**
Let $r > 0$ and $0 \leq a \leq b$. For any random variable Y, prove that

$$a^r P(|Y| \geq a) + (b^r - a^r)P(|Y| \geq b) \leq \mathbb{E}[|Y|^r].$$

10. **Order Statistics and Stochastic Geometry**

(a) (**Optional-Extra Credit**) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed continuous random variables with common density function f. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the *order statistics* corresponding to these random variables, i.e., $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the ordered values of $X_1, X_2, \ldots, X_n$. Argue that

$$f_{X_{(1)}, X_{(2)}, \ldots, X_{(n)}}(x_1, x_2, \ldots, x_n) = n! \prod_{i=1}^{n} f(x_i), \quad x_1 \leq x_2 \leq \cdots \leq x_n.$$

(b) Consider the $[0, 1]$ interval. Assume that three points are distributed at random over this interval, i.e., their positions are independent and uniformly distributed over this interval. Use part (a) to find the probability that any two points are at distance $\geq d$ apart, where $d \leq 1/2$.